CHAPTER 10

Structural Dynamics

10.1 INTRODUCTION

In addition to static analyses, the finite element method is a powerful tool for analyzing the dynamic response of structures. As illustrated in Chapter 7, the finite element method in combination with the finite difference method can be used to examine the transient response of heat transfer situations. A similar approach can be used to analyze the transient dynamic response of mechanical structures. However, in the analysis of structures, an additional tool is available. The tool, known as *modal analysis,* has its basis in the fact that every mechanical structure exhibits natural modes of vibration (dynamic response) and these modes can be readily computed given the elastic and inertia characteristics of the structure.

In this chapter, we introduce the concept of natural modes of vibration via the simple harmonic oscillator system. Using the finite element concepts developed in earlier chapters, the simple harmonic oscillator is represented as a finite element system and the basic ideas of natural frequency and natural mode are introduced. The single degree of freedom simple harmonic oscillator is then extended to multiple degrees of freedom, to illustrate the existence of multiple natural frequencies and vibration modes. From this basis, we proceed to more general dynamic analyses using the finite element method.

10.2 THE SIMPLE HARMONIC OSCILLATOR

The so-called simple harmonic oscillator is a combination of a linear elastic spring having free length L and a concentrated mass as shown in Figure 10.1a. The mass of the spring is considered negligible. The system is assumed to be subjected to gravity in the vertical direction, and the upper end of the spring is attached to a rigid support. With the system in equilibrium as in Figure 10.1b, the

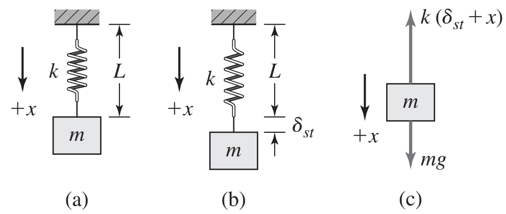

Figure 10.1
(a) Simple harmonic oscillator. (b) Static equilibrium position. (c) Free-body diagram for arbitrary position *x*.

gravitational force is in equilibrium with the spring force so

$$\sum F_x = 0 = mg - k\delta_{st} \tag{10.1}$$

where δ_{st} is the equilibrium elongation of the spring and x is measured positive downward from the equilibrium position; that is, when $x = 0$, the system is at its equilibrium position.

If, by some action, the mass is displaced from its equilibrium position, the force system becomes unbalanced, as shown by the free-body diagram of Figure 10.1c. We must apply Newton's second law to obtain

$$\sum F_x = ma_x = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = mg - k(\delta_{st} + x) \tag{10.2}$$

Incorporating the equilibrium condition expressed by Equation 10.1, Equation 10.2 becomes

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0 \tag{10.3}$$

Equation 10.3 is a second-order, linear, ordinary differential equation with constant coefficients. (And physically, we assume that the coefficients m and k are positive.) Equation 10.3 is most-often expressed in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0 \tag{10.4}$$

The general solution for Equation 10.4 is

$$x(t) = A\sin\omega t + B\cos\omega t \tag{10.5}$$

where A and B are the constants of integration. Recall that the solution of a second-order differential equation requires the specification of two constants to determine the solution to a specific problem. When the differential equation describes the time response of a mechanical system, the constants of integration are most-often called the *initial conditions.*

Equation 10.5 shows that the variation of displacement of the mass as a function of time is periodic. Using basic trigonometric identities, Equation 10.5 can be equivalently expressed as

$$x(t) = C\sin(\omega t + \phi) \tag{10.6}$$

where the constants A and B have been replaced by constants of integration C and ϕ. Per Equation 10.6, the mass oscillates sinusoidally at *circular frequency* ω and with constant *amplitude C*. Phase angle ϕ is indicative of position at time 0 since $x(0) = C\sin\phi$. Also, note that, since $x(t)$ is measured about the equilibrium position, the oscillation occurs about that position. The circular frequency is

$$\omega = \sqrt{\frac{k}{m}}\ \text{rad/sec} \tag{10.7}$$

and is a constant value determined by the physical characteristics of the system. In this simple case, the *natural circular frequency,* as it is often called, depends on the spring constant and mass only. Therefore, if the mass is displaced from the equilibrium position and released, the oscillatory motion occurs at a constant frequency determined by the physical parameters of the system. In the case described, the oscillatory motion is described as *free vibration,* since the system is free of all external forces excepting gravitational attraction.

Next, we consider the simple harmonic oscillator in the finite element context. From Chapter 2, the stiffness matrix of the spring is

$$\left[k^{(e)}\right] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{10.8}$$

and the equilibrium equations for the element are

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{10.9}$$

which is identical to Equation 2.4. However, the spring element is not in static equilibrium, so we must examine the nodal forces in detail.

Figure 10.2 shows free-body diagrams of the spring element and mass, respectively. The free-body diagrams depict snapshots in time when the system is in motion and, hence, are dynamic free-body diagrams. As the mass of the spring is considered negligible, Equation 10.9 is valid for the spring element. For the mass, we have

$$\sum F_x = ma_x = m\frac{\mathrm{d}^2u_2}{\mathrm{d}t^2} = mg - f_2 \tag{10.10}$$

from which the force on node 2 is

$$f_2 = mg - m\frac{\mathrm{d}^2u_2}{\mathrm{d}t^2} \tag{10.11}$$

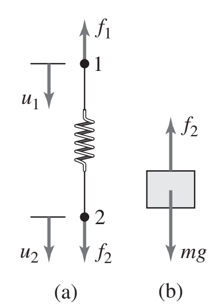

Figure 10.2 Free-body diagrams of (a) a spring and (b) a mass, when treated as parts of a finite element system.

Substituting for f_2 in Equation 10.9 gives

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ mg - m\ddot{u}_2 \end{Bmatrix} \tag{10.12}$$

where $\ddot{u}_2 = \mathrm{d}^2u_2/\mathrm{d}t^2$. The dynamic effect of the inertia of the attached mass is shown in the second of the two equations represented by Equation 10.12. Equation 10.12 can also be expressed as

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ mg \end{Bmatrix} \tag{10.13}$$

where we have introduced the mass matrix

$$[m] = \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \tag{10.14}$$

and the nodal acceleration matrix

$$\{\ddot{u}\} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} \tag{10.15}$$

For the simple harmonic oscillator of Figure 10.1, we have the constraint (boundary) condition $u_1 = 0$, so the first of Equation 10.13 becomes simply $-ku_2 = f_1$, while the second equation is

$$m\ddot{u}_2 + ku_2 = mg \tag{10.16}$$

Note that Equation 10.16 is *not* the same as Equation 10.3. Do the two equations represent the same physical phenomenon? To show that the answer is yes, we solve Equation 10.16 and compare the results with the solution given in Equation 10.6.

Recalling that the solution of any differential equation is the sum of a homogeneous (complementary) solution and a particular solution, both solutions must be obtained for Equation 10.16, since the equation is not homogeneous (i.e., the right-hand side is nonzero). Setting the right-hand side to zero, the form of the homogeneous equation is the same as that of Equation 10.3, so by analogy, the homogeneous solution is

$$u_{2h}(t) = C\sin(\omega t + \phi) \tag{10.17}$$

where ω, C, and ϕ are as previously defined. The particular solution must satisfy Equation 10.16 exactly for all values of time. As the right-hand side is constant, the particular solution must also be constant; hence,

$$u_{2p}(t) = \frac{mg}{k} = \delta_{st} \tag{10.18}$$

which represents the static equilibrium solution per Equation 10.1. The complete solution is then

$$u_2(t) = u_{2h}(t) + u_{2p}(t) = \delta_{st} + C\sin(\omega t + \phi) \tag{10.19}$$

Equation 10.19 represents a sinusoidal oscillation around the equilibrium position and is, therefore, the same as the solution given in Equation 10.6. Given the displacement of node 2, the reaction force at node 1 is obtained via the constraint equation as

$$f_1 = -ku_2(t) = -k(\delta_{st} + C\sin(\omega t + \phi)) \tag{10.20}$$

Amplitude C and phase angle ϕ are determined by application of the initial conditions, as illustrated in the following example.

EXAMPLE 10.1

A simple harmonic oscillator has $k = 25$ lb/in. and $mg = 20$ lb. The mass is displaced downward a distance of 1.5 in. from the equilibrium position. The mass is released from that position with zero initial velocity at $t = 0$. Determine (a) the natural circular frequency, (b) the amplitude of the oscillatory motion, and (c) the phase angle of the oscillatory motion.

■ **Solution**

The natural circular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{25}{20/386.4}} = 21.98 \text{ rad/sec}$$

where, for consistency of units, the mass is obtained from the weight using $g = 386.4$ in./s^2.

The given initial conditions are

$$u_2(t = 0) = \delta_{st} + 1.5 \text{ in.} \qquad \dot{u}_2(t = 0) = 0 \text{ in./sec}$$

and the static deflection is $\delta_{st} = W/k = 20/25 = 0.8$ in. Therefore, we have $u_2(0) = 2.3$ in. The motion of node 2 (hence, the mass) is then given by Equation 10.19 as

$$u_2(t) = 0.8 + C\sin(21.98t + \phi) \text{ in.}$$

and the velocity is

$$\dot{u}_2(t) = \frac{\mathrm{d}u_2}{\mathrm{d}t} = 21.98C\cos(21.98t + \phi) \text{ in./sec}$$

Applying the initial conditions results in the equations

$$u_2(t = 0) = 2.3 = 0.8 + C\sin\phi$$
$$\dot{u}_2(t) = 0 = 21.98C\cos\phi$$

The initial velocity equation is satisfied by $C = 0$ or $\phi = \pi/2$. If the former is true, the initial displacement equation cannot be satisfied, so we conclude that $\phi = \pi/2$. Substituting into the displacement equation then gives the amplitude C as 1.5 in. The complete motion solution is

$$u_2(t) = 0.8 + 1.5\sin\left(21.98t + \frac{\pi}{2}\right) = 0.8 + 1.5\cos(21.98t) \text{ in.}$$

indicating that the mass oscillates 1.5 in. above and below the static equilibrium position continuously in time and completes one cycle every $2\pi/21.98$ sec. Therefore, the cyclic frequency is

$$f = \frac{\omega}{2\pi} = \frac{21.98}{2\pi} = 3.5 \text{ cycles/sec (Hz)}$$

The cyclic frequency is often simply referred to as the *natural frequency.* The time required to complete one cycle of motion is known as the *period* of oscillation, given by

$$\tau = \frac{1}{f} = \frac{1}{3.5} = 0.286 \text{ sec}$$

10.2.1 Forced Vibration

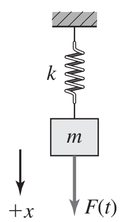

Figure 10.3 Simple harmonic oscillator subjected to external force $F(t)$.

Figure 10.3 shows a simple harmonic oscillator in which the mass is acted on by a time-varying external force $F(t)$. The resulting motion is known as *forced vibration,* owing to the presence of the external forcing function. As the only difference in the applicable free-body diagrams is the external force acting on the mass, the finite element form of the system equations can be written directly from Equation 10.13 as

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ mg + F(t) \end{Bmatrix} \tag{10.21}$$

While the constraint equation for the reaction force at node 1 is unchanged, the differential equation for the motion of node 2 is now

$$m\ddot{u}_2 + ku_2 = mg + F(t) \tag{10.22}$$

The complete solution for Equation 10.22 is the sum of the homogeneous solution and two particular solutions, since two nonzero terms are on the right-hand side. As we already obtained the homogeneous solution and the particular solution for the *mg* term, we focus on the particular solution for the external force. The particular solution of interest must satisfy

$$m\ddot{u}_2 + ku_2 = F(t) \tag{10.23}$$

exactly for all values of time. Dividing by the mass, we obtain

$$\ddot{u}_2 + \omega^2 u_2 = \frac{F(t)}{m} \tag{10.24}$$

where $\omega^2 = k/m$ is the square of the natural circular frequency. Of particular importance in structural dynamic analysis is the case when external forcing functions exhibit sinusoidal variation in time, since such forces are quite common. Therefore, we consider the case in which

$$F(t) = F_0 \sin \omega_f t \tag{10.25}$$

where F_0 is the amplitude or maximum value of the force and ω_f is the circular frequency of the forcing function, or *forcing frequency* for short. Equation 10.24 becomes

$$\ddot{u}_2 + \omega^2 u_2 = \frac{F_0}{m} \sin \omega_f t \tag{10.26}$$

To satisfy Equation 10.24 exactly for all values of time, the terms on the left must contain a sine function identical to the sine term on the right-hand side. Since the second derivative of the sine function is another sine function, we assume a solution in the form $u_2(t) = U \sin \omega_f t$, where U is a constant to be determined. Differentiating twice and substituting, Equation 10.26 becomes

$$-U\omega_f^2 \sin \omega_f t + U\omega^2 \sin \omega_f t = \frac{F_0}{m} \sin \omega_f t \tag{10.27}$$

from which

$$U = \frac{F_0/m}{\omega^2 - \omega_f^2} \tag{10.28}$$

The particular solution representing response of the simple harmonic oscillator to a sinusoidally varying force is then

$$u_2(t) = \frac{F_0/m}{\omega^2 - \omega_f^2} \sin \omega_f t \tag{10.29}$$

The motion represented by Equation 10.29 is most often simply called the *forced response* and exhibits two important characteristics: (1) the frequency of the forced response is the same as the frequency of the forcing function, and (2) if the circular frequency of the forcing function is very near the natural circular frequency of the system, the denominator in Equation 10.29 becomes very small. The latter is an extremely important observation, as the result is large amplitude of motion. In the case $\omega_f = \omega$, Equation 10.29 indicates an infinite amplitude. This condition is known as *resonance,* and for this reason, the natural circular frequency of the system is often called the *resonant frequency*. Mathematically, Equation 10.29 is not a valid solution for the resonant condition (Problem 10.5); however, the correct solution for the resonant condition nevertheless exhibits unbounded amplitude growth with time.

The simple harmonic oscillator just modeled contains no device for energy dissipation (*damping*). Consequently, the free vibration solution, Equation 10.20, represents motion that continues without end. Physically, such motion is not possible, since all systems contain some type of dissipation mechanism, such as internal or external friction, air resistance, or devices specifically designed for the purpose. Similarly, the infinite amplitude indicated for the resonant condition cannot be attained by a real system because of the presence of damping. However, relatively large, yet bounded, amplitudes occur at or near the resonant frequency. Hence, the resonant condition is to be avoided if at all possible. As is subsequently shown, physical systems actually exhibit multiple natural frequencies, so multiple resonant conditions exist.

10.3 MULTIPLE DEGREES-OF-FREEDOM SYSTEMS

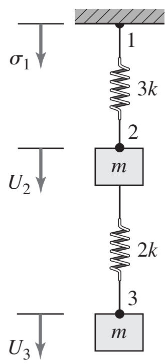

Figure 10.4 A spring-mass system exhibiting 2 degrees of freedom.

Figure 10.4 shows a system of two spring elements having concentrated masses attached at nodes 2 and 3 in the global coordinate system. As in previous examples, the system is subjected to gravity and the upper spring is attached to a rigid support at node 1. Of interest here is the dynamic response of the system of two springs and two masses when the equilibrium condition is disturbed by some external influence and then free to oscillate without external force. We could take the Newtonian mechanics approach by drawing the appropriate free-body diagrams and applying Newton's second law of motion to obtain the governing equations. Instead, we take the finite element approach. By now, the procedure of assembling the system stiffness matrix should be routine. Following the procedure, we obtain

$$[K] = \begin{bmatrix} 3k & -3k & 0 \\ -3k & 5k & -2k \\ 0 & -2k & 2k \end{bmatrix} \tag{10.30}$$

as the system stiffness matrix. But what of the mass/inertia matrix? As the masses are concentrated at element nodes, we define the system mass matrix as

$$[M] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{10.31}$$

The equations of motion can be expressed as

$$[M]\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + [K]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ mg \\ mg \end{Bmatrix} \tag{10.32}$$

where R_1 is the dynamic reaction force at node 1.

Invoking the constraint condition $U_1 = 0$, Equation 10.32 become

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} mg \\ mg \end{Bmatrix} \tag{10.33}$$

which is a system of two second-order, linear, ordinary differential equations in the two unknown system displacements U_2 and U_3. As the gravitational forces indicated by the forcing function represent the static equilibrium condition, these are neglected and the system of equations rewritten as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.34}$$

As a practical matter, most finite element software packages do *not* include the structural weight in an analysis problem. Instead, inclusion of the structural weight is an option that must be selected by the user of the software. Whether to include gravitational effects is a judgment made by the analyst based on the specifics of a given structural geometry and loading.

The system of second-order, linear, ordinary, homogeneous differential equations given by Equation 10.34 represents the free-vibration response of the 2 degrees-of-freedom system of Figure 10.4. As a freely oscillating system, we seek solutions in the form of harmonic motion as

$$\begin{aligned} U_2(t) &= A_2 \sin(\omega t + \phi) \\ U_3(t) &= A_3 \sin(\omega t + \phi) \end{aligned} \tag{10.35}$$

where A_2 and A_3 are the vibration amplitudes of nodes 2 and 3 (the masses attached to nodes 2 and 3); ω is an unknown, assumed harmonic circular frequency of motion; and ϕ is the phase angle of such motion. Taking the second derivatives with respect to time of the assumed solutions and substituting into Equation 10.34 results in

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} \sin(\omega t + \phi) + \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} \sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.36}$$

or

$$\begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} \sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.37}$$

Equation 10.37 is a system of two, homogeneous algebraic equations, which must be solved for the vibration amplitudes A_2 and A_3. From linear algebra, a system of homogeneous algebraic equations has nontrivial solutions if and only if the determinant of the coefficient matrix is zero. Therefore, for nontrivial solutions,

$$\begin{vmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{vmatrix} = 0 \tag{10.38}$$

which gives

$$(5k - m\omega^2)(2k - m\omega^2) - 4k^2 = 0 \tag{10.39}$$

Equation 10.39 is known as the *characteristic equation* or *frequency equation* of the physical system. As k and m are known positive constants, Equation 10.39 is treated as a quadratic equation in the unknown ω^2 and solved by the quadratic formula to obtain *two* roots

$$\begin{aligned} \omega_1^2 &= \frac{k}{m} \\ \omega_2^2 &= 6\frac{k}{m} \end{aligned} \tag{10.40}$$

or

$$\omega_1 = \sqrt{\frac{k}{m}}$$
$$\omega_2 = \sqrt{6\frac{k}{m}} \tag{10.41}$$

In mathematical rigor, there are four roots, since the negative values corresponding to Equation 10.41 also satisfy the frequency equation. The negative values are rejected because a negative frequency has no physical meaning and use of the negative values in the assumed solution (Equation 10.35) introduces only a phase shift and represents the same motion as that corresponding to the positive root.

The 2 degrees-of-freedom system of Figure 10.4 is found to have two natural circular frequencies of oscillation. As is customary, the numerically smaller of the two is designated as ω_1 and known as the *fundamental* frequency. The task remains to determine the amplitudes A_2 and A_3 in the assumed solution. For this purpose, Equation 10.37 is

$$\begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.42}$$

As Equation 10.42 is a set of homogeneous equations, we can find no absolute values of the amplitudes. We can, however, obtain information regarding the numerical relations among the amplitudes as follows. If we substitute $\omega^2 = \omega_1^2 = k/m$ into *either* algebraic equation, we obtain $A_3 = 2A_2$, which defines the *amplitude ratio* $A_3/A_2 = 2$ for the first, or fundamental, mode of vibration. That is, if the system oscillates at its fundamental frequency ω_1, the amplitude of oscillation of m_2 is twice that of m_1. (Note that we are unable to calculate the absolute value of either amplitude; only the ratio can be determined. The absolute values depend on the initial conditions of motion, as is subsequently illustrated.) The displacement equations for the fundamental mode are then

$$\begin{aligned} U_2(t) &= A_2^{(1)} \sin(\omega_1 t + \phi_1) \\ U_3(t) &= A_3^{(1)} \sin(\omega_1 + \phi_1) = 2A_2^{(1)} \sin(\omega_1 t + \phi_1) \end{aligned} \tag{10.43}$$

where the superscript on the amplitudes is used to indicate that the displacements correspond to vibration at the fundamental frequency.

Next we substitute the second natural circular frequency $\omega^2 = \omega_2^2 = 6k/m$ into either equation and obtain the relation $A_3 = -0.5A_2$, which defines the second amplitude ratio as $A_3/A_2 = -0.5$. So, in the second natural mode of vibration, the masses move in opposite directions. The displacements corresponding to the second frequency are then

$$\begin{aligned} U_2(t) &= A_2^{(2)} \sin(\omega_2 t + \phi_2) \\ U_3(t) &= A_3^{(2)} \sin(\omega_2 + \phi_2) = -0.5A_2^{(2)} \sin(\omega_2 t + \phi_2) \end{aligned} \tag{10.44}$$

where again the superscript refers to the frequency.

Therefore, the free-vibration response of the 2 degree-of-freedom system is given by

$$\begin{aligned} U_2(t) &= A_2^{(1)}\sin(\omega_1 t + \phi_1) + A_2^{(2)}\sin(\omega_2 t + \phi_2) \\ U_3(t) &= 2A_2^{(1)}\sin(\omega_1 t + \phi_1) - 0.5A_2^{(2)}\sin(\omega_2 + \phi_2) \end{aligned} \tag{10.45}$$

and we note the four unknown constants in the solution; specifically, these are the amplitudes $A_2^{(1)}$, $A_2^{(2)}$ and the phase angles ϕ_1 and ϕ_2. Evaluation of the constants is illustrated in a subsequent example.

Depending on the reader's mathematical background, the analysis of the 2 degree-of-freedom vibration problem may be recognized as an *eigenvalue problem* [1]. The computed natural circular frequencies are the eigenvalues of the problem and the amplitude ratios represent the eigenvectors of the problem. Equation 10.45 represents the response of the system in terms of the natural modes of vibration. Such a solution is often referred to as being obtained by *modal superposition* or simply *modal analysis*. To represent the complete solution for the system, we use the matrix notation

$$\begin{Bmatrix} U_2(t) \\ U_3(t) \end{Bmatrix} = \begin{Bmatrix} A_2^{(1)} \\ 2A_2^{(1)} \end{Bmatrix} \sin(\omega_1 t + \phi_1) + \begin{Bmatrix} A_2^{(2)} \\ -0.5A_2^{(2)} \end{Bmatrix} \sin(\omega_2 t + \phi_2) \tag{10.46}$$

which shows that the modes interact to produce the overall motion of the system.

EXAMPLE 10.2

Given the system of Figure 10.4 with $k = 40$ lb/in. and $mg = W = 20$ lb, determine

(a) The natural frequencies of the system.
(b) The free response, if the initial conditions are

$$U_2(t=0) = 1 \text{ in.} \qquad U_3(t=0) = 0.5 \text{ in.} \qquad \dot{U}_2(t=0) = \dot{U}_3(t=0) = 0$$

These initial conditions are specified in reference to the equilibrium position of the system, so the computed displacement functions do not include the effect of gravity.

■ Solution

Per Equation 10.41, the natural circular frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{40}{20/g}} = \sqrt{\frac{40(386.4)}{20}} = 27.8 \text{ rad/sec}$$

$$\omega_2 = \sqrt{\frac{6k}{m}} = \sqrt{\frac{6(40)}{20/g}} = \sqrt{\frac{6(40)(386.4)}{20}} = 68.1 \text{ rad/sec}$$

The free-vibration response is given by Equation 10.35 as

$$\begin{aligned} U_2(t) &= A_2^{(1)}\sin(27.8t + \phi_1) + A_2^{(2)}\sin(68.1t + \phi_2) \\ U_3(t) &= 2A_2^{(1)}\sin(27.8t + \phi_1) - 0.5A_2^{(2)}\sin(68.1t + \phi_2) \end{aligned}$$

The amplitudes and phase angles are determined by applying the initial conditions, which are

$$U_2(0) = 1 = A_2^{(1)} \sin\phi_1 + A_2^{(2)} \sin\phi_2$$
$$U_3(0) = 0.5 = 2A_2^{(1)} \sin\phi_1 - 0.5A_2^{(2)} \sin\phi_2$$
$$\dot{U}_2(0) = 0 = 27.8A_2^{(1)} \cos\phi_1 + 68.1A_2^{(2)} \cos\phi_2$$
$$\dot{U}_3(0) = 0 = 2(27.8)A_2^{(1)} \cos\phi_1 - 0.5(68.1)A_2^{(2)} \cos\phi_2$$

The initial conditions produce a system of four algebraic equations in the four unknowns $A_2^{(1)}$, $A_2^{(2)}$, ϕ_1, ϕ_2. Solution of the equations is not trivial, owing to the presence of the trigonometric functions. Letting $P = A_2^{(1)} \sin\phi_1$ and $Q = A_2^{(2)} \sin\phi_2$, the displacement initial condition equations become

$$P + Q = 1$$
$$2P - 0.5Q = 0.5$$

which are readily solved to obtain

$$P = A_2^{(1)} \sin\phi_1 = 0.4 \quad \text{and} \quad Q = A_2^{(2)} \sin\phi_2 = 0.6$$

Similarly, setting $R = A_2^{(1)} \cos\phi_1$ and $S = A_2^{(2)} \sin\phi_2$, the initial velocity equations are

$$27.8R + 68.1S = 0$$
$$2(27.8)R - 0.5(68.1)S = 0$$

representing a homogeneous system in the variables R and S. Nontrivial solutions exist only if the determinant of the coefficient matrix is zero. In this case, the determinant is not zero, as may easily be verified by direct computation. There are no nontrivial solutions; hence, $R = S = 0$. Based on physical argument, the amplitudes cannot be zero, so we must conclude that $\cos\phi_1 = \cos\phi_2 = 0 \Rightarrow \phi_1 = \phi_2 = \pi/2$. It follows that the sine function of the phase angles have unity value; hence, $A_2^{(1)} = 0.4$ and $A_2^{(2)} = 0.6$. Substituting the amplitudes into the general solution form while noting that $\sin(\omega t + \pi/2) = \cos\omega t$, the free-vibration response of each mass is

$$U_2(t) = 0.4 \cos 27.8t + 0.6 \cos 68.1t$$
$$U_3(t) = 0.8 \cos 27.8t - 0.3 \cos 68.1t$$

The displacement response of each mass is seen to be a combination of motions corresponding to the natural circular frequencies of the system. Such a phenomenon is characteristic of vibrating structural systems. All the natural modes of vibration participate in the general motion of a structure.

10.3.1 Many-Degrees-of-Freedom Systems

As illustrated by the system of two springs and masses, there are two natural frequencies and two natural modes of vibration. If we extend the analysis to a system of springs and masses having N degrees of freedom, as depicted in Figure 10.5, and apply the assembly procedure for a finite element analysis, the finite element equations are of the form

$$[M]\{\ddot{U}\} + [K]\{U\} = \{0\} \tag{10.47}$$

where $[M]$ is the system mass matrix and $[K]$ is the system stiffness matrix. To determine the natural frequencies and mode shapes of the system's vibration modes, we assume, as in the 1 and 2 degrees-of-freedom cases, that

$$U_i(t) = A_i \sin(\omega t + \phi) \tag{10.48}$$

Substitution of the assumed solution into the system equations leads to the frequency equation

$$|[K] - \omega^2[M]| = 0 \tag{10.49}$$

which is a polynomial of order N in the variable ω^2. The solution of Equation 10.49 results in N natural frequencies ω_j, which, for structural systems, can be shown to be real but not necessarily distinct; that is, repeated roots can occur. As discussed many times, the finite element equations cannot be solved unless boundary conditions are applied so that the equations become inhomogeneous. A similar phenomenon exists when determining the system natural frequencies and mode shapes. If the system is not constrained, rigid body motion is possible and one or more of the computed natural frequencies has a value of zero. A three-dimensional system has six zero-valued natural frequencies, corresponding to rigid body translation in the three coordinate axes and rigid body rotations about the three coordinate axes. Therefore, if improperly constrained, a structural system exhibits repeated zero roots of the frequency equation.

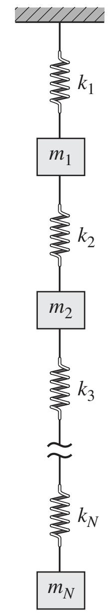

Figure 10. 5 A spring-mass system exhibiting arbitrarily many degrees of freedom.

Assuming that constraints are properly applied, the frequencies resulting from the solution of Equation 10.49 are substituted, one at a time, into Equation 10.47 and the amplitude ratios (eigenvectors) computed for each natural mode of vibration. The general solution for each degree of freedom is then expressed as

$$U_i(t) = \sum_{j=1}^{N} A_i^{(j)} \sin(\omega_j t + \phi_j) \qquad i = 1, N \tag{10.50}$$

illustrating that the displacement of each mass is the sum of contributions from each of the N natural modes. Displacement solutions expressed by Equation 10.50 are said to be obtained by *modal superposition.* We add the independent solutions of the linear differential equations of motion.

EXAMPLE 10.3

Determine the natural frequencies and modal amplitude vectors for the 3 degrees-of-freedom system depicted in Figure 10.6a.

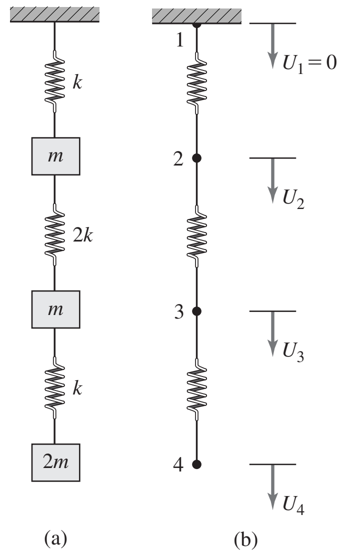

Figure 10.6 System with 3 degrees of freedom for Example 10.3.

■ Solution

The finite element model is shown in Figure 10.6b, with node and element numbers as indicated. Assembly of the global stiffness matrix results in

$$[K] = \begin{bmatrix} k & -k & 0 & 0 \\ -k & 3k & -2k & 0 \\ 0 & -2k & 3k & -k \\ 0 & 0 & -k & k \end{bmatrix}$$

Similarly, the assembled global mass matrix is

$$[M] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 2m \end{bmatrix}$$

Owing to the constraint $U_1 = 0$, we need consider only the last three equations of motion, given by

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{Bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming sinusoidal response as $U_i = A_i \sin(\omega t + \phi)$, $i = 2, 4$ and substituting into the equations of motion leads to the frequency equation

$$\begin{vmatrix} 3k - \omega^2 m & -2k & 0 \\ -2k & 3k - \omega^2 m & -k \\ 0 & -k & k - 2\omega^2 m \end{vmatrix} = 0$$

Expanding the determinant and simplifying gives

$$\omega^6 - 6.5\frac{k}{m}\omega^4 + 7.5\left(\frac{k}{m}\right)^2\omega^2 - \left(\frac{k}{m}\right)^3 = 0$$

which will be treated as a cubic equation in the unknown ω^2. Setting $\omega^2 = C(k/m)$, the frequency equation becomes

$$(C^3 - 6.5C^2 + 7.5C - 1)\left(\frac{k}{m}\right)^3 = 0$$

which has the roots

$$C_1 = 0.1532 \qquad C_2 = 1.2912 \qquad C_3 = 5.0556$$

The corresponding natural circular frequencies are

$$\omega_1 = 0.3914\sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.1363\sqrt{\frac{k}{m}}$$

$$\omega_3 = 2.2485\sqrt{\frac{k}{m}}$$

To obtain the amplitude ratios, we substitute the natural circular frequencies into the amplitude equations one at a time while setting (arbitrarily) $A_2^{(i)} = 1$, $i = 1, 2, 3$ and solve for the amplitudes $A_3^{(i)}$ and $A_4^{(i)}$. Using ω_1 results in

$$\begin{aligned}
\left(3k - \omega_1^2 m\right)A_2^{(1)} - 2kA_3^{(1)} &= 0\\
-2kA_2^{(1)} + \left(3k - \omega_1^2 m\right)A_3^{(1)} - kA_4^{(1)} &= 0\\
-kA_3^{(1)} + \left(k - 2\omega_1^2 m\right)A_4^{(1)} &= 0
\end{aligned}$$

Substituting $\omega_1 = 0.3914\sqrt{k/m}$, we obtain

$$\begin{aligned}
2.847A_2^{(1)} - 2A_3^{(1)} &= 0\\
-2A_2^{(1)} + 2.847A_3^{(1)} - A_4^{(1)} &= 0\\
-A_3^{(1)} + 0.694A_4^{(1)} &= 0
\end{aligned}$$

As discussed, the amplitude equations are homogeneous; explicit solutions cannot be obtained. We can, however, determine the amplitude ratios by setting $A_2^{(1)} = 1$ to obtain

$$A_3^{(1)} = 1.4235$$
$$A_4^{(1)} = 2.0511$$

The amplitude vector corresponding to the fundamental mode ω_1 is then represented as

$$\left\{A^{(1)}\right\} = A_2^{(1)}\begin{Bmatrix} 1 \\ 1.4325 \\ 2.0511 \end{Bmatrix}$$

and this is the *eigenvector* corresponding to the *eigenvalue* ω_1. Proceeding identically with the values for the other two frequencies, ω_2 and ω_3, the resulting amplitude vectors are

$$\left\{A^{(2)}\right\} = A_2^{(2)} \begin{Bmatrix} 1 \\ 0.8544 \\ -0.5399 \end{Bmatrix}$$

$$\left\{A^{(3)}\right\} = A_2^{(3)} \begin{Bmatrix} 1 \\ -1.0279 \\ 0.1128 \end{Bmatrix}$$

This example illustrates that an N degree-of-freedom system exhibits N natural modes of vibration defined by N natural circular frequencies and the corresponding N amplitude vectors (mode shapes). While the examples deal with discrete spring-mass systems, where the motions of the masses are easily visualized as recognizable events, structural systems modeled via finite elements exhibit N natural frequencies and N mode shapes, where N is the number of degrees of freedom (displacements in structural systems) represented by the finite element model. Accuracy of the computed frequencies as well as use of the natural modes of vibration to examine response to external forces is delineated in following sections.

10.4 BAR ELEMENTS: CONSISTENT MASS MATRIX

In the preceding discussions of spring-mass systems, the mass (inertia) matrix in each case is a lumped (diagonal) matrix, since each mass is directly attached to an element node. In these simple cases, we neglect the mass of the spring elements in comparison to the concentrated masses. In the general case of solid structures, the mass is distributed geometrically throughout the structure and the inertia properties of the structure depend directly on the mass distribution. To illustrate the effects of distributed mass, we first consider longitudinal (axial) vibration of the bar element of Chapter 2.

The bar element shown in Figure 10.7a is the same as the bar element introduced in Chapter 2 with the very important difference that displacements and applied forces are now assumed to be time dependent, as indicated. The free-body diagram of a differential element of length dx is shown in Figure 10.7b, where cross-sectional area A is assumed constant. Applying Newton's second law to the differential element gives

$$\left(\sigma + \frac{\partial \sigma}{\partial x}\,\mathrm{d}x\right)A - \sigma A = (\rho A\,\mathrm{d}x)\frac{\partial^2 u}{\partial t^2} \tag{10.51}$$

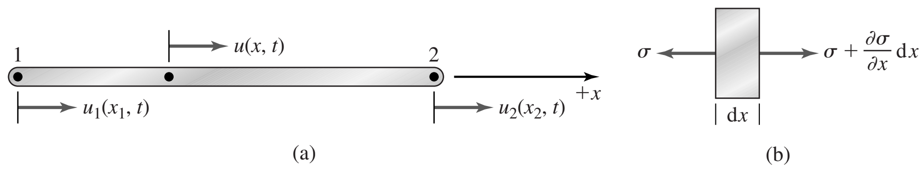

Figure 10.7
(a) Bar element exhibiting time-dependent displacement. (b) Free-body diagram of a differential element.

where ρ is density of the bar material. Note the use of partial derivative operators, since displacement is now considered to depend on both position and time. Substituting the stress-strain relation $\sigma = E\varepsilon = E(\partial u/\partial x)$, Equation 10.51 becomes

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} \tag{10.52}$$

Equation 10.52 is the *one-dimensional wave equation,* the governing equation for propagation of elastic displacement waves in the axial bar.

In the dynamic case, the axial displacement is discretized as

$$u(x, t) = N_1(x)u_1(t) + N_2(x)u_2(t) \tag{10.53}$$

where the nodal displacements are now expressed explicitly as time dependent, but the interpolation functions remain dependent only on the spatial variable. Consequently, the interpolation functions are identical to those used previously for equilibrium situations involving the bar element: $N_1(x) = 1 - (x/L)$ and $N_2(x) = x/L$. Application of Galerkin's method to Equation 10.52 in analogy to Equation 5.29 yields the residual equations as

$$\int_0^L N_i(x)\left(E\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2}\right)A\ \mathrm{d}x = 0 \qquad i = 1, 2 \tag{10.54}$$

Assuming constant material properties, Equation 10.54 can be written as

$$\rho A\int_0^L N_i(x)\frac{\partial^2 u}{\partial t^2}\,\mathrm{d}x = AE\int_0^L N_i(x)\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x \qquad i = 1, 2 \tag{10.55}$$

Mathematical treatment of the right-hand side of Equation 10.55 is identical to that presented in Chapter 5 and is not repeated here, other than to recall that the result of the integration and combination of the two residual equations in matrix form is

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \Rightarrow [k]\{u\} = \{f\} \tag{10.56}$$

Substituting the discretized approximation for $u(x, t)$, the integral on the left becomes

$$\rho A \int_0^L N_i(x) \frac{\partial^2 u}{\partial t^2}\, dx = \rho A \int_0^L N_i(N_1 \ddot{u}_1 + N_2 \ddot{u}_2)\, dx \qquad i = 1, 2 \tag{10.57}$$

where the double-dot notation indicates differentiation with respect to time. The two equations represented by Equation 10.57 are written in matrix form as

$$\rho A \int_0^L \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} dx \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = \frac{\rho A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = [m]\{\ddot{u}\} \tag{10.58}$$

and the reader is urged to confirm the result by performing the indicated integrations. Also note that the mass matrix is symmetric but not singular. Equation 10.58 defines the *consistent* mass matrix for the bar element. The term *consistent* is used because the interpolation functions used in formulating the mass matrix are the same as (consistent with) those used to describe the spatial variation of displacement. Combining Equations 10.56 and 10.58 per Equation 10.55, we obtain the dynamic finite element equations for a bar element as

$$\frac{\rho A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{10.59}$$

or

$$[m]\{\ddot{u}\} + [k]\{u\} = \{f\} \tag{10.60}$$

and we note that $\rho AL = m$ is the total mass of the element. (Why is the sign of the second term positive?)

Given the governing equations, let us now determine the natural frequencies of a bar element in axial vibration. Per the foregoing discussion of free vibration, we set the nodal force vector to zero and write the frequency equation as

$$|[k] - \omega^2[m]| = 0 \tag{10.61}$$

to obtain

$$\begin{vmatrix} k - \omega^2 \dfrac{m}{3} & -\left(k + \omega^2 \dfrac{m}{6}\right) \\ -\left(k + \omega^2 \dfrac{m}{6}\right) & k - \omega^2 \dfrac{m}{3} \end{vmatrix} = 0 \tag{10.62}$$

Expanding Equation 10.62 results in a quadratic equation in ω^2

$$\left(k - \omega^2 \frac{m}{3}\right)^2 - \left(k + \omega^2 \frac{m}{6}\right)^2 = 0 \tag{10.63}$$

or

$$\omega^2\left(\omega^2 - 12\frac{k}{m}\right) = 0 \tag{10.64}$$

Equation 10.64 has roots $\omega^2 = 0$ and $\omega^2 = 12k/m$. The zero root arises because we specify no constraint on the element; hence, rigid body motion is possible and represented by the zero-valued natural circular frequency. The nonzero natural circular frequency corresponds to axial displacement waves in the bar, which could occur, for example, if the free bar were subjected to an axial impulse at one end. In such a case, rigid body motion would occur but axial vibration would simultaneously occur with circular frequency $\omega_1 = \sqrt{12k/m} = (3.46/L)\sqrt{E/\rho}$. The following example illustrates determination of natural circular frequencies for a constrained bar.

EXAMPLE 10.4

Using two equal-length finite elements, determine the natural circular frequencies of the solid circular shaft fixed at one end shown in Figure 10.8a.

■ **Solution**

The elements and node numbers are shown in Figure 10.8b. The characteristic stiffness of each element is

$$k = \frac{AE}{L/2} = \frac{2AE}{L}$$

so that the element stiffness matrices are

$$\left[k^{(1)}\right] = \left[k^{(2)}\right] = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The mass of each element is

$$m = \frac{\rho AL}{2}$$

and the element consistent mass matrices are

$$\left[m^{(1)}\right] = \left[m^{(2)}\right] = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

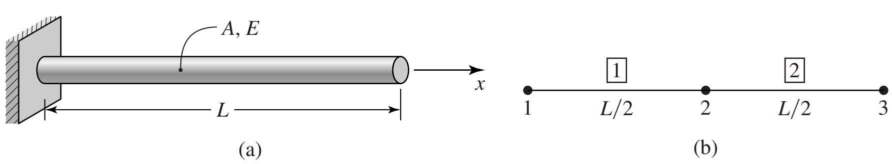

Figure 10.8
(a) Circular shaft of Example 10.4. (b) Model using two bar elements.

Following the direct assembly procedure, the global stiffness matrix is

$$[K] = \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and the global consistent mass matrix is

$$[M] = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The global equations of motion are then

$$\frac{\rho AL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Applying the constraint condition $U_1 = 0$, we have

$$\frac{\rho AL}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \frac{2AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

as the homogeneous equations governing free vibration. For convenience, the last equation is rewritten as

$$\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \frac{24E}{\rho L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming sinusoidal responses

$$U_2 = A_2 \sin(\omega t + \phi) \qquad U_3 = A_3 \sin(\omega t + \phi)$$

differentiating twice and substituting results in

$$-\omega^2\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix}\sin(\omega t + \phi) + \frac{24E}{\rho L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix}\sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Again, we obtain a set of homogeneous algebraic equations that have nontrivial solutions only if the determinant of the coefficient matrix is zero. Letting $\lambda = 24E/\rho L^2$, the frequency equation is given by the determinant

$$\begin{vmatrix} 2\lambda - 4\omega^2 & -\lambda - \omega^2 \\ -\lambda - \omega^2 & \lambda - 2\omega^2 \end{vmatrix} = 0$$

which, when expanded and simplified, is

$$7\omega^4 - 10\lambda\omega^2 + \lambda^2 = 0$$

Treating the frequency equation as a quadratic in ω^2, the roots are obtained as

$$\omega_1^2 = 0.1082\lambda \qquad \omega_2^2 = 1.3204\lambda$$

Substituting for λ, the natural circular frequencies are

$$\omega_1 = \frac{1.611}{L}\sqrt{\frac{E}{\rho}} \qquad \omega_2 = \frac{5.629}{L}\sqrt{\frac{E}{\rho}} \text{ rad/sec}$$

For comparison purposes, we note that the exact solution [2] for the natural circular frequencies of a bar in axial vibration yields the fundamental natural circular frequency as $1.571/L\sqrt{E/\rho}$ and the second frequency as $4.712/L\sqrt{E/\rho}$. Therefore, the error for the first computed frequency is about 2.5 percent, while the error in the second frequency is about 19 percent.

It is also informative to note (see Problem 10.12) that, if the lumped mass matrix approach is used for this example, we obtain

$$\omega_1 = \frac{1.531}{L}\sqrt{\frac{E}{\rho}} \qquad \omega_2 = \frac{3.696}{L}\sqrt{\frac{E}{\rho}} \text{ rad/sec}$$

The solution for Example 10.4 yielded two natural circular frequencies for free axial vibration of a bar fixed at one end. Such a bar has an infinite number of natural frequencies, like any element or structure having continuously distributed mass. In finite element modeling, the partial differential equations governing motion of continuous systems are discretized into a finite number of algebraic equations for approximate solutions. Hence, the number of frequencies obtainable via a finite element approach is limited by the discretization inherent to the finite element model.

The inertia characteristics of a bar element can also be represented by a lumped mass matrix, similar to the approach used in the spring-mass examples earlier in this chapter. In the lumped matrix approach, half the total mass of the element is assumed to be concentrated at each node and the connecting material is treated as a massless spring with axial stiffness. The lumped mass matrix for a bar element is then

$$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{10.65}$$

Use of lumped mass matrices offers computational advantages. Since the element mass matrix is diagonal, assembled global mass matrices also are diagonal. On the other hand, although more computationally difficult in use, consistent mass matrices can be proven to provide upper bounds for the natural circular frequencies [3]. No such proof exists for lumped matrices. Nevertheless, lumped mass matrices are often used, particularly with bar and beam elements, to obtain reasonably accurate predictions of dynamic response.

10.5 BEAM ELEMENTS

We now develop the mass matrix for a beam element in flexural vibration. First, the consistent mass matrix is obtained using an approach analogous to that for the bar element in the previous section. Figure 10.9 depicts a differential element of a beam in flexure under the assumption that the applied loads are time dependent. As the situation is otherwise the same as that of Figure 5.3 except for the use of

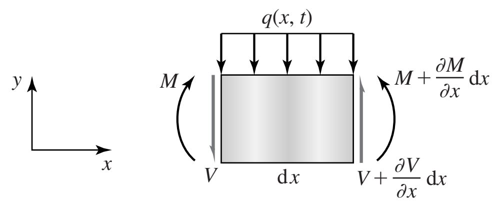

Figure 10.9 Differential element of a beam subjected to time-dependent loading.

partial derivatives, we apply Newton's second law of motion to the differential element in the y direction to obtain

$$\sum F_y = ma_y \Rightarrow V + \frac{\partial V}{\partial x}\,\mathrm{d}x - V - q(x,t)\,\mathrm{d}x = (\rho A\,\mathrm{d}x)\frac{\partial^2 v}{\partial t^2} \qquad (10.66)$$

where ρ is the material density and A is the cross-sectional area of the element. The quantity ρA represents mass per unit length in the x direction. Equation 10.66 simplifies to

$$\frac{\partial V}{\partial x} - q(x,t) = \rho A\frac{\partial^2 v}{\partial t^2} \qquad (10.67)$$

As we are dealing with the small deflection theory of beam flexure, beam slopes, therefore rotations, are small. Therefore, we neglect the rotational inertia of the differential beam element and apply the moment equilibrium equation. The result is identical to that of Equation 5.37, repeated here as

$$\frac{\partial M}{\partial x} = -V \qquad (10.68)$$

Substituting the moment-shear relation into Equation 10.67 gives

$$-\frac{\partial^2 M}{\partial x^2} - q(x,t) = \rho A\frac{\partial^2 v}{\partial t^2} \qquad (10.69)$$

Finally, the flexure formula

$$M = EI_z\frac{\partial^2 v}{\partial x^2} \qquad (10.70)$$

is substituted into Equation 10.69 to obtain the governing equation for dynamic beam deflection as

$$-\frac{\partial^2}{\partial x^2}\left(EI_z\frac{\partial^2 v}{\partial x^2}\right) - q(x,t) = \rho A\frac{\partial^2 v}{\partial t^2} \qquad (10.71)$$

Under the assumptions of constant elastic modulus E and moment of inertia I_z, the governing equation becomes

$$\rho A\frac{\partial^2 v}{\partial t^2} + EI_z\frac{\partial^4 v}{\partial x^4} = -q(x,t) \qquad (10.72)$$

As in the case of the bar element, transverse beam deflection is discretized using the same interpolation functions previously developed for the beam function. Now, however, the nodal displacements are assumed to be time dependent. Hence,

$$v(x,t) = N_1(x)v_1(t) + N_2(x)\theta_1(t) + N_3(x)v_2(t) + N_4(x)\theta_2(t) \tag{10.73}$$

and the interpolation functions are as given in Equation 4.26 or 4.29. Application of Galerkin's method to Equation 10.72 for a finite element of length L results in the residual equations

$$\int_0^L N_i(x)\left(\rho A \frac{\partial^2 v}{\partial t^2} + EI_z \frac{\partial^4 v}{\partial x^4} + q\right) = 0 \qquad i = 1, 4 \tag{10.74}$$

As the last two terms of the integrand are the same as treated in Equation 5.42, development of the stiffness matrix and nodal force vector are not repeated here. Instead, we focus on the first term of the integrand, which represents the terms of the mass matrix.

For each of the four equations represented by Equation 10.74, the first integral term becomes

$$\rho A \int_0^L N_i(N_1\ddot{v}_1 + N_2\ddot{\theta}_1 + N_3\ddot{v}_2 + N_4\ddot{\theta}_2)\,\mathrm{d}x = \rho A \int_0^L N_i[N]\,\mathrm{d}x \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix} \qquad i = 1, 4 \tag{10.75}$$

and, when all four equations are expressed in matrix form, the inertia terms become

$$\rho A \int_0^L [N]^T[N]\,\mathrm{d}x \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix} = \left[m^{(e)}\right] \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix} \tag{10.76}$$

The consistent mass matrix for a two-dimensional beam element is given by

$$\left[m^{(e)}\right] = \rho A \int_0^L [N]^T[N]\,\mathrm{d}x \tag{10.77}$$

Substitution for the interpolation functions and performing the required integrations gives the mass matrix as

$$\left[m^{(e)}\right] = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \tag{10.78}$$

and it is to be noted that we have assumed constant cross-sectional area in this development.

Combining the mass matrix with previously obtained results for the stiffness matrix and force vector, the finite element equations of motion for a beam element are

$$\left[m^{(e)}\right]\begin{Bmatrix}\ddot{v}_1\\ \ddot{\theta}_1\\ \ddot{v}_2\\ \ddot{\theta}_2\end{Bmatrix} + \left[k^{(e)}\right]\begin{Bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{Bmatrix} = -\int_0^L [N]^T q(x,t)\,\mathrm{d}x + \begin{Bmatrix}-V_1(t)\\ -M_1(t)\\ V_2(t)\\ M_2(t)\end{Bmatrix} \tag{10.79}$$

and all quantities are as previously defined. In the dynamic case, the nodal shear forces and bending moments may be time dependent, as indicated.

Assembly procedures for the beam element including the mass matrix are identical to those for the static equilibrium case. The global mass matrix is directly assembled, using the individual element mass matrices in conjunction with the element-to-global displacement relations. While system assembly is procedurally straightforward, the process is tedious when carried out by hand. Consequently, a complex example is not attempted. Instead, a relatively simple example of natural frequency determination is examined.

EXAMPLE 10.5

Using a single finite element, determine the natural circular frequencies of vibration of a cantilevered beam of length L, assuming constant values of ρ, E, and A.

■ **Solution**

The beam is depicted in Figure 10.10, with node 1 at the fixed support such that the boundary (constraint) conditions are $v_1 = \theta_1 = 0$. For free vibration, applied force and bending moment at the free end (node 2) are $V_2 = M_2 = 0$ and there is no applied distributed load. Under these conditions, the first two equations represented by Equation 10.79 are constraint equations and not of interest. Using the constraint conditions and the known applied forces, the last two equations are

$$\frac{\rho AL}{420}\begin{bmatrix}156 & -22L\\ -22L & 4L^2\end{bmatrix}\begin{Bmatrix}\ddot{v}_2\\ \ddot{\theta}_2\end{Bmatrix} + \frac{EI_z}{L^3}\begin{bmatrix}12 & -6L\\ -6L & 4L^2\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

For computational convenience, the equations are rewritten as

$$\begin{bmatrix}156 & -22L\\ -22L & 4L^2\end{bmatrix}\begin{Bmatrix}\ddot{v}_2\\ \ddot{\theta}_2\end{Bmatrix} + \frac{420EI_z}{mL^3}\begin{bmatrix}12 & -6L\\ -6L & 4L^2\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

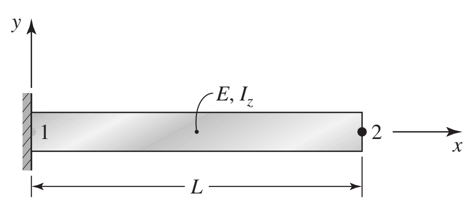

Figure 10.10 The cantilevered beam of Example 10.5 modeled as one element.

with $m = \rho AL$ representing the total mass of the beam. Assuming a sinusoidal displacement response, the frequency equation becomes

$$\begin{vmatrix} 12\lambda - 156\omega^2 & -6\lambda L + 22\omega^2 L \\ -6\lambda L + 22\omega^2 L & 4L^2(\lambda - \omega^2) \end{vmatrix} = 0$$

with $\lambda = 420EI_z/mL^3$. After expanding the determinant and performing considerable algebraic manipulation, the frequency equation becomes

$$5\omega^4 - 102\lambda\omega^2 + 3\lambda^2 = 0$$

Solving as a quadratic in ω^2, the roots are

$$\omega_1^2 = 0.02945\lambda \qquad \omega_2^2 = 20.37\lambda$$

Substituting for λ in terms of the beam physical parameters, we obtain

$$\omega_1 = 3.517\sqrt{\frac{EI_z}{mL^3}} \qquad \omega_2 = 92.50\sqrt{\frac{EI_z}{mL^3}} \text{ rad/sec}$$

as the finite element approximations to the first two natural circular frequencies. For comparison, the exact solution gives

$$\omega_1^{\text{exact}} = 3.516\sqrt{\frac{EI_z}{mL^3}} \qquad \omega_2^{\text{exact}} = 22.03\sqrt{\frac{EI_z}{mL^3}} \text{ rad/sec}$$

The fundamental frequency computed via a single element is essentially the same as the exact solution, whereas the second computed frequency is considerably larger than the corresponding exact value. As noted previously, a continuous system exhibits an infinite number of natural modes; we obtained only two modes in this example. If the number of elements is increased, the number of frequencies (natural modes) that can be computed increases as the number of degrees of freedom increases. In concert, the accuracy of the computed frequencies improves.

If the current example is refined by using two elements having length $L/2$ and the solution procedure repeated, we can compute four natural frequencies, the lowest two given by

$$\omega_1 = 3.516\sqrt{\frac{EI_z}{mL^3}} \qquad \omega_2 = 24.5\sqrt{\frac{EI_z}{mL^3}}$$

and we observe that the second natural circular frequency has improved (in terms of the exact solution) significantly. The third and fourth frequencies from this solution are found to be quite high in relation to the known exact values.

As indicated by the foregoing example, the number of natural frequencies and mode shapes that can be computed depend directly on the number of degrees of freedom of the finite element model. Also, as would be expected for convergence, as the number of degrees of freedom increases, the computed frequencies become closer to the exact values. As a general rule, the lower values (numerically) converge more rapidly to exact solution values. While this is discussed in more detail in conjunction with specific examples to follow, a general rule of thumb for frequency analysis is as follows: If the finite element analyst is interested in the first P modes of vibration of a structure, at least $2P$ modes should be calculated. Note that this implies the capability of calculating a subset of frequencies rather than all frequencies of a model. Indeed, this is possible and extremely important, since a practical finite element model may have thousands of degrees of freedom, hence thousands of natural frequencies. The computational burden of calculating all the frequencies is overwhelming and unnecessary, as is discussed further in the following section.

10.6 MASS MATRIX FOR A GENERAL ELEMENT: EQUATIONS OF MOTION

The previous examples dealt with relatively simple systems composed of linear springs and the bar and beam elements. In these cases, direct application of Newton's second law and Galerkin's finite element method led directly to the formulation of the matrix equations of motion; hence, the element mass matrices. For more general structural elements, an energy-based approach is preferred, as for static analyses. The approach to be taken here is based on *Lagrangian mechanics* and uses an energy method based loosely on *Lagrange's equations of motion* [4].

Prior to examining a general case, we consider the simple harmonic oscillator of Figure 10.1. At an arbitrary position x with the system assumed to be in motion, kinetic energy of the mass is

$$T = \frac{1}{2}m\dot{x}^2 \tag{10.80}$$

and the total potential energy is

$$U_e = \frac{1}{2}k(\delta_{st} + x)^2 - mg(\delta_{st} + x) \tag{10.81}$$

therefore, the total mechanical energy is

$$E_m = T + U_e = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(\delta_{st} + x)^2 - mg(\delta_{st} + x) \tag{10.82}$$

As the simple harmonic oscillator model contains no mechanism for energy removal, the principle of conservation of mechanical energy applies; hence,

$$\frac{\mathrm{d}E_m}{\mathrm{d}t} = 0 = m\dot{x}\ddot{x} + k(\delta_{st} + x)\dot{x} - mg\dot{x} \tag{10.83}$$

or

$$m\ddot{x} + k(\delta_{st} + x) = mg \tag{10.84}$$

and the result is exactly the same as obtained via Newton's second law in Equation 10.2.

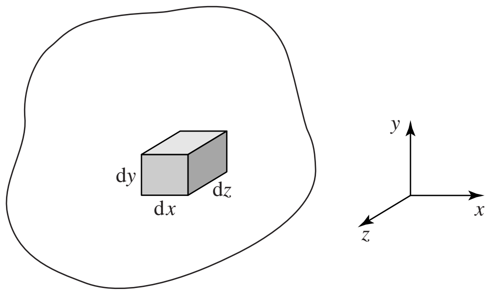

Figure 10.11 Differential element of a general three-dimensional body.

For the general case, we consider the three-dimensional body depicted in Figure 10.11 and examine a differential mass $\mathrm{d}m = \rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ located at arbitrary position (x, y, z). Displacement of the differential mass in the coordinate directions are (u, v, w) and the velocity components are $(\dot{u}, \dot{v}, \dot{w})$, respectively. As we previously examined the potential energy, we now focus on kinetic energy of the differential mass given by

$$\mathrm{d}T = \frac{1}{2}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,\mathrm{d}m = \frac{1}{2}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{10.85}$$

Total kinetic energy of the body is then

$$T = \frac{1}{2}\iiint(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,\mathrm{d}m = \frac{1}{2}\iiint(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{10.86}$$

and the integration is performed over the entire mass (volume) of the body.

Considering the body to be a finite element with the displacement field discretized as

$$\begin{aligned} u(x, y, z, t) &= \sum_{i=1}^{M} N_i(x, y, z)u_i(t) = [N]\{u\} \\ v(x, y, z, t) &= \sum_{i=1}^{M} N_i(x, y, z)v_i(t) = [N]\{v\} \\ w(x, y, z, t) &= \sum_{i=1}^{M} N_i(x, y, z)w_i(t) = [N]\{w\} \end{aligned} \tag{10.87}$$

(where M is the number of element nodes), the velocity components can be expressed as

$$\begin{aligned} \dot{u} &= \frac{\partial u}{\partial t} = [N]\{\dot{u}\} \\ \dot{v} &= \frac{\partial v}{\partial t} = [N]\{\dot{v}\} \\ \dot{w} &= \frac{\partial w}{\partial t} = [N]\{\dot{w}\} \end{aligned} \tag{10.88}$$

The element kinetic energy expressed in terms of nodal velocities and interpolation functions is then written as

$$T^{(e)} = \frac{1}{2} \iiint\limits_{V^{(e)}} (\{\dot{u}\}^T [N]^T [N]\{\dot{u}\} + \{\dot{v}\}^T [N]^T [N]\{\dot{v}\} + \{\dot{w}\}^T [N]^T [N]\{\dot{w}\})\rho \,\mathrm{d}V^{(e)} \tag{10.89}$$

Denoting the nodal velocities as

$$\{\dot{\delta}\} = \begin{Bmatrix} \{\dot{u}\} \\ \{\dot{v}\} \\ \{\dot{w}\} \end{Bmatrix} \tag{10.90}$$

a $3M \times 1$ column matrix, the kinetic energy is expressed as

$$\begin{aligned} T^{(e)} &= \frac{1}{2}\{\dot{\delta}\}^T \iiint\limits_{V^{(e)}} \begin{bmatrix} [N]^T[N] & 0 & 0 \\ 0 & [N]^T[N] & 0 \\ 0 & 0 & [N]^T[N] \end{bmatrix} \rho \,\mathrm{d}V^{(e)} \{\dot{\delta}\} \\ &= \frac{1}{2}\{\dot{\delta}\}^T \left[m^{(e)}\right]\{\dot{\delta}\} \end{aligned} \tag{10.91}$$

and the element mass matrix is thus identified as

$$\left[m^{(e)}\right] = \iiint\limits_{V^{(e)}} \begin{bmatrix} [N]^T[N] & 0 & 0 \\ 0 & [N]^T[N] & 0 \\ 0 & 0 & [N]^T[N] \end{bmatrix} \rho \,\mathrm{d}V^{(e)} \tag{10.92}$$

Note that, in Equation 10.92, the zero terms actually represent $M \times M$ null matrices. Therefore, the mass matrix as derived is a $3M \times 3M$ matrix, which is also readily shown to be symmetric. Also note that the mass matrix of Equation 10.92 is a *consistent* mass matrix. The following example illustrates the computations for a two-dimensional element.

EXAMPLE 10.6

Formulate the mass matrix for the two-dimensional rectangular element depicted in Figure 10.12. The element has uniform thickness 5 mm and density $\rho = 7.83 \times 10^{-6}$ kg/mm^3.

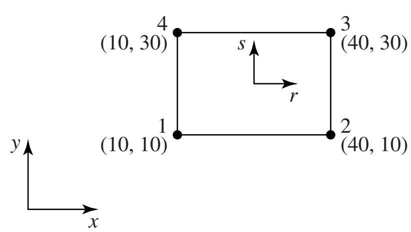

Figure 10.12 The rectangular element of Example 10.6.

■ **Solution**

Per Equation 6.56, the interpolation functions in terms of serendipity or natural coordinates are

$$N_1(r,s) = \frac{1}{4}(1-r)(1-s)$$

$$N_2(r,s) = \frac{1}{4}(1+r)(1-s)$$

$$N_3(r,s) = \frac{1}{4}(1+r)(1+s)$$

$$N_4(r,s) = \frac{1}{4}(1-r)(1+s)$$

with $r = (x-25)/15$ and $s = (y-20)/10$. For integration in the natural coordinates, $\mathrm{d}x = 15\,\mathrm{d}r$ and $\mathrm{d}y = 10\,\mathrm{d}s$. The mass matrix is 8×8 and the nonzero terms are defined by

$$\iiint\limits_{V^{(e)}} [N]^T[N]\rho\,\mathrm{d}V^{(e)} = \rho t \int_{-1}^{1}\int_{-1}^{1} [N]^T[N](15\,\mathrm{d}r)(10\,\mathrm{d}s)$$

$$= 150(5)\rho \int_{-1}^{1}\int_{-1}^{1} [N]^T[N]\,\mathrm{d}r\,\mathrm{d}s$$

In this solution, we compute a few terms for illustration, then present the overall results. For example,

$$m_{11} = 150(5)\rho \int_{-1}^{1}\int_{-1}^{1} N_1^2\,\mathrm{d}r\,\mathrm{d}s = \frac{150(5)}{16}\rho \int_{-1}^{1}\int_{-1}^{1} (1-r)^2(1-s)^2\,\mathrm{d}r\,\mathrm{d}s$$

$$= \frac{150(5)}{16}\rho\left[\frac{(1-r)^3}{3}\frac{(1-s)^3}{3}\right]_{-1}^{1} = \frac{750}{16}\rho\left(\frac{64}{9}\right) = \frac{4(750)}{9}(7.830)(10)^{-6}$$

$$= 2.6(10)^{-3}\ \text{kg}$$

Similarly,

$$m_{12} = 150(5)\rho \int_{-1}^{1}\int_{-1}^{1} N_1N_2\,\mathrm{d}r\,\mathrm{d}s = \frac{150(5)}{16}\rho \int_{-1}^{1}\int_{-1}^{1} (1-r^2)(1-s)^2\,\mathrm{d}r\,\mathrm{d}s$$

$$= \frac{150(5)}{16}\rho\left[\left(r-\frac{r^3}{3}\right)\left(\frac{(1-s)^3}{3}\right)(-1)\right]_{-1}^{1}$$

$$= \frac{150(5)}{16}(7.83)(10)^{-6}\left(\frac{32}{9}\right) = 1.3(10)^{-3}\ \text{kg}$$

If we carry out all the integrations indicated to form the mass matrix, the final result for the rectangular element is

$$\left[m^{(e)}\right] = \begin{bmatrix} 2.6 & 1.3 & 0.7 & 1.3 & 0 & 0 & 0 & 0 \\ 1.3 & 2.6 & 1.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0.7 & 1.3 & 2.6 & 1.3 & 0 & 0 & 0 & 0 \\ 1.3 & 0.7 & 1.3 & 2.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.6 & 1.3 & 0.7 & 1.3 \\ 0 & 0 & 0 & 0 & 1.3 & 2.6 & 1.3 & 0.7 \\ 0 & 0 & 0 & 0 & 0.7 & 1.3 & 2.6 & 1.3 \\ 0 & 0 & 0 & 0 & 1.3 & 0.7 & 1.3 & 2.6 \end{bmatrix} (10)^{-3} \text{ kg}$$

We observe that the element mass matrix is symmetric, as expected. Also note that storing the entire matrix as shown would be quite inefficient, since only the 4×4 submatrix of nonzero terms is needed.

Having developed a general formulation for the mass matrix of a finite element, we return to the determination of the equations of motion of a structure modeled via the finite element method and subjected to dynamic (that is, time-dependent) loading. If we have in hand, as we do, the mass and stiffness matrices of a finite element, we can assemble the global equations for a finite element model of a structure and obtain an expression for the total energy in the form

$$\frac{1}{2}\{\dot{q}\}^T[M]\{\dot{q}\} + \frac{1}{2}\{q\}^T[K]\{q\} - \{q\}^T\{f\} = E \tag{10.93}$$

where $\{q\}$ is the column matrix of displacements described in the global coordinate system and all other terms are as previously defined. (At this point, we reemphasize that Equation 10.93 models the response of an ideal elastic system, which contains no mechanism for energy dissipation.) For a system as described, total mechanical energy is constant, so that $\mathrm{d}E/\mathrm{d}t = 0$. As the mechanical energy is expressed as a function of both velocity and displacement, the minimization procedure requires that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial t} + \frac{\partial E}{\partial q_i}\frac{\partial q_i}{\partial t} = 0 \qquad i = 1, P \tag{10.94}$$

where we now represent the total number of degrees of freedom of the model as P to avoid confusion with the mass matrix notation $[M]$. Application of Equation 10.94 to the energy represented by Equation 10.93 yields a system of ordinary differential equations

$$[M]\{\ddot{q}\} + [K]\{q\} = \{F\} \tag{10.95}$$

Equation 10.94 is not necessarily mathematically rigorous in every case. However, for the systems under consideration, in which there is no energy removal mechanism and the total potential energy includes the effect of external forces, the resulting equations of motion are the same as those given by both the *Lagrangian* approach and *variational principles* [5].

Examination of Equation 10.95 in light of known facts about the stiffness and mass matrices reveals that the differential equations are coupled, at least through the stiffness matrix, which is known to be symmetric but not diagonal. The phenomena embodied here is referred to as *elastic coupling,* as the coupling terms arise from the elastic stiffness matrix. In consistent mass matrices, the equations are also coupled by the nondiagonal nature of the mass matrix; therefore, the term *inertia coupling* is applied when the mass matrix is not diagonal. Obtaining solutions for coupled differential equations is not generally a straightforward prodecure. We show, however, that the modal characteristics embodied in the equations of motion can be used to advantage in examining system response to harmonic (sinusoidal) forcing functions. The so-called *harmonic response* is a capability of essentially any finite element software package, and the general techniques are discussed in the following section, after a brief discussion of natural modes.

In the absence of externally applied nodal forces, Equation 10.95 is a system of P homogeneous, linear second-order differential equations in the independent variable time. Hence, we have an eigenvalue problem in which the eigenvalues are the natural circular frequencies of oscillation of the structural system, and the eigenvectors are the amplitude vectors (mode shapes) corresponding to the natural frequencies. The frequency equation is represented by the determinant

$$|-\omega^2[M] + [K]| = 0 \tag{10.96}$$

If formally expanded, this determinant yields a polynomial of order P in the variable ω^2. Solution of the frequency polynomial results in computation of P natural circular frequencies and P modal amplitude vectors. The free-vibration response of such a system is then described by the sum (superposition) of the natural vibration modes as

$$\delta_i(t) = \sum_{j=1}^{P} A_i^{(j)} \sin(\omega_j t + \phi_j) \qquad i = 1, P \tag{10.97}$$

Note that the superposition indicated by Equation 10.97 is valid only for linear differential equations.

In Equation 10.97, the $A_i^{(j)}$ and ϕ_j are to be determined to satisfy given initial conditions. In accord with previous examples for simpler systems, we know that the amplitude vectors for a given modal frequency can be determined within a single unknown constant, so we can write the modal amplitude vectors as

$$\left\{A^{(i)}\right\} = A_1^{(i)} \begin{Bmatrix} 1 \\ \beta_2^{(i)} \\ \beta_3^{(i)} \\ \vdots \\ \beta_P^{(i)} \end{Bmatrix} \qquad i = 1, P \tag{10.98}$$

where the β terms are known constants resulting from substitution of the natural circular frequencies into the governing equations for the amplitudes. For a system having P degrees of freedom, we have $2P$ unknown constants $A_1^{(i)}$ and ϕ_i, $i = 1, P$ in the motion solution. The constants are determined by application of $2P$ initial conditions, which are generally specified as the displacements and velocities of the nodes at time $t = 0$. While the natural modes of free vibration are important in and of themselves, application of modal analysis to the harmonically forced response of structural systems is a very important concept. Prior to examination of the forced response, we derive a very important property of the principal vibration modes.

10.7 ORTHOGONALITY OF THE PRINCIPAL MODES

The principal modes of vibration of systems with multiple degrees of freedom share a fundamental mathematical property known as *orthogonality*. The free-vibration response of a multiple degrees-of-freedom system is described by Equation 10.95 with $\{F\} = 0$ as

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\} \tag{10.99}$$

Assuming that we have solved for the natural circular frequencies and the modal amplitude vectors via the assumed solution form $q_i(t) = A_i \sin(\omega t + \phi)$, substitution of a particular frequency ω_i into Equation 10.99 gives

$$-\omega_i^2[M]\{A^{(i)}\} + [K]\{A^{(i)}\} = 0 \tag{10.100}$$

and for any other frequency ω_j

$$-\omega_j^2[M]\{A^{(j)}\} + [K]\{A^{(j)}\} = 0 \tag{10.101}$$

Multiplying Equation 10.100 by $\{A^{(j)}\}^T$ and Equation 10.101 by $\{A^{(i)}\}^T$ gives

$$-\omega_i^2\{A^{(j)}\}^T[M]\{A^{(i)}\} + \{A^{(j)}\}^T[K]\{A^{(i)}\} = 0 \tag{10.102}$$

$$-\omega_j^2\{A^{(i)}\}^T[M]\{A^{(j)}\} + \{A^{(i)}\}^T[K]\{A^{(j)}\} = 0 \tag{10.103}$$

Subtracting Equation 10.102 from Equation 10.103, we have

$$\{A^{(j)}\}^T[M]\{A^{(i)}\}\left(\omega_i^2 - \omega_j^2\right) = 0 \qquad i \neq j \tag{10.104}$$

In arriving at the result represented by Equation 10.104, we utilize the fact from matrix algebra that $[A]^T[B][C] = [C]^T[B][A]$, where $[A]$, $[B]$, $[C]$ are any three matrices for which the triple product is defined. As the two circular frequencies in Equation 10.104 are distinct, we conclude that

$$\{A^{(j)}\}^T[M]\{A^{(i)}\} = 0 \qquad i \neq j \tag{10.105}$$

Equation 10.105 is the mathematical statement of *orthogonality of the principal modes of vibration*. The orthogonality property provides a very powerful mathematical technique for decoupling the equations of motion of a multiple degrees-of-freedom system.

For a system exhibiting P degrees of freedom, we define the *modal matrix* as a $P \times P$ matrix in which the columns are the amplitude vectors for each natural mode of vibration; that is,

$$[A] = \left[\left\{ A^{(1)} \right\} \left\{ A^{(2)} \right\} \ldots \left\{ A^{(P)} \right\} \right] \tag{10.106}$$

and consider the matrix triple product $[S] = [A]^T[M][A]$. Per the orthogonality condition, Equation 10.105, each off-diagonal term of the matrix represented by the triple product is zero; hence, the matrix $[S] = [A]^T[M][A]$ is a *diagonal* matrix. The diagonal (nonzero) terms of the matrix have magnitude

$$S_{ii} = \left\{ A^{(i)} \right\}^T [M] \left\{ A^{(i)} \right\} \qquad i = 1, P \tag{10.107}$$

As each modal amplitude vector is known only within a constant multiple (recall in earlier examples that we set $A_1^{(i)} = 1$ arbitrarily), the modal amplitude vectors can be manipulated such that the diagonal terms described by Equation 10.107 can be made to assume any desired numerical value. In particular, if the value is selected as unity, so that

$$S_{ii} = \left\{ A^{(i)} \right\}^T [M] \left\{ A^{(i)} \right\} = 1 \qquad i = 1, P \tag{10.108}$$

then the modal amplitude vectors are said to be *orthonormal* and the matrix triple product becomes

$$[S] = [A]^T[M][A] = [I] \tag{10.109}$$

where $[I]$ is the $P \times P$ identity matrix.

Normalizing the modal amplitude vectors per Equation 10.108 is a straightforward procedure, as follows. Let a specific modal amplitude be represented by Equation 10.98 in which the first term is arbitrarily assigned value of unity. The corresponding diagonal term of the modal matrix is then

$$\sum_{j=1}^{P} \sum_{k=1}^{P} m_{jk} A_j^{(i)} A_k^{(i)} = S_{ii} = \text{constant} \tag{10.110}$$

If we redefine the terms of the modal amplitude vector so that

$$A_j^{(i)} = \frac{A_j^{(i)}}{\sqrt{S_{ii}}} = \frac{A_j^{(i)}}{\sqrt{\displaystyle\sum_{j=1}^{P} \sum_{k=1}^{P} m_{jk} A_j^{(i)} A_k^{(i)}}} \qquad i = 1, P \tag{10.111}$$

the matrix described by Equation 10.109 is indeed the identity matrix.

Having established the orthogonality concept and normalized the modal matrix, we return to the general problem described by Equation 10.95, in which the force vector is no longer assumed to be zero. For reasons that will become apparent, we introduce the change of variables

$$\{q\} = [A]\{p\} \tag{10.112}$$

where $\{p\}$ is the column matrix of *generalized displacements,* which are linear combinations of the actual nodal displacements $\{q\}$, and $[A]$ is the normalized modal matrix. Equation 10.95 then becomes

$$[M][A]\{\ddot{p}\} + [K][A]\{p\} = \{F\} \tag{10.113}$$

Premultiplying by $[A]^T$, we obtain

$$[A]^T[M][A]\{\ddot{p}\} + [A]^T[K][A]\{p\} = [A]^T\{F\} \tag{10.114}$$

Utilizing the orthogonality principle, Equation 10.114 is

$$[I]\{\ddot{p}\} + [A]^T[k][A]\{p\} = [A]^T\{F\} \tag{10.115}$$

Now we must examine the stiffness effects as represented by $[A]^T[K][A]$. Given that [K] is a symmetric matrix, the triple product $[A]^T[K][A]$ is also a symmetric matrix. Following the previous development of orthogonality of the principal modes, the triple product $[A]^T[K][A]$ is also easily shown to be a diagonal matrix. The values of the diagonal terms are found by multiplying Equation 10.100 by $\{A^{(i)}\}^T$ to obtain

$$-\omega_i^2\{A^{(i)}\}^T[M]\{A^{(i)}\} + \{A^{(i)}\}^T[K]\{A^{(i)}\} = 0 \qquad i = 1, P \tag{10.116}$$

If the modal amplitude vectors have been normalized as described previously, Equation 10.116 is

$$\{A^{(i)}\}^T[K]\{A^{(i)}\} = \omega_i^2 \qquad i = 1, P \tag{10.117}$$

hence, the matrix triple product $[A]^T[K][A]$ produces a diagonal matrix having diagonal terms equal to the squares of the natural circular frequencies of the principal modes of vibration; that is,

$$[A]^T[K][A] = \begin{bmatrix} \omega_1^2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \omega_2^2 & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & & \omega_P^2 \end{bmatrix} \tag{10.118}$$

Finally, Equation 10.115 becomes

$$[I]\{\ddot{p}\} + [\omega^2]\{p\} = [A]^T\{F\} \tag{10.119}$$

with matrix $[\omega^2]$ representing the diagonal matrix defined in Equation 10.118.

EXAMPLE 10.7

Using the data of Example 10.3, normalize the modal matrix and verify that $[A]^T[M][A] = [I]$ and $[A]^T[K][A] = [\omega^2]$.

■ **Solution**

For the first mode, we have

$$S_{11} = \{A^{(1)}\}^T[M]\{A^{(1)}\} = [1 \quad 1.4325 \quad 2.0511]\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{Bmatrix} 1 \\ 1.4325 \\ 2.0511 \end{Bmatrix}$$
$$= 11.4404\,m$$

so the first modal amplitude vector is normalized by dividing each term by $\sqrt{S_{11}} = 3.3824\sqrt{m}$, which gives the normalized vector as

$$\{A^{(1)}\} = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.2956 \\ 0.4289 \\ 0.6064 \end{Bmatrix}$$

Applying the same procedure to the modal amplitude vectors for the second and third modes gives

$$\{A^{(2)}\} = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.6575 \\ 0.5618 \\ -0.3550 \end{Bmatrix} \qquad \{A^{(3)}\} = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.6930 \\ -0.7124 \\ 0.0782 \end{Bmatrix}$$

and the normalized modal matrix is

$$[A] = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.2956 & 0.6575 & 0.6930 \\ 0.4289 & 0.5618 & -0.7124 \\ 0.6064 & -0.3550 & 0.0782 \end{bmatrix}$$

To verify Equation 10.109, we form the triple product

$$[A]^T[M][A] = \frac{1}{m}\begin{bmatrix} 0.2956 & 0.4289 & 0.6064 \\ 0.6575 & 0.5618 & -0.3550 \\ 0.6930 & -0.7124 & 0.0782 \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$
$$\times\begin{bmatrix} 0.2956 & 0.6575 & 0.6930 \\ 0.4289 & 0.5618 & -0.7124 \\ 0.6064 & -0.3550 & 0.0782 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as expected.

The triple product with respect to the stiffness matrix is

$$[A]^T[K][A] = \frac{k}{m}\begin{bmatrix} 0.2956 & 0.4289 & 0.6064 \\ 0.6575 & 0.5618 & -0.3550 \\ 0.6990 & -0.7124 & 0.0782 \end{bmatrix}\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$
$$\times\begin{bmatrix} 0.2956 & 0.6575 & 0.6990 \\ 0.4289 & 0.5618 & -0.7124 \\ 0.6064 & -0.3550 & 0.0782 \end{bmatrix}$$

which evaluates to

$$[A]^T[K][A] = \frac{k}{m}\begin{bmatrix} 0.1532 & 0 & 0 \\ 0 & 1.2912 & 0 \\ 0 & 0 & 5.0557 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}$$

10.8 HARMONIC RESPONSE USING MODE SUPERPOSITION

The orthogonality condition of the principal modes is especially useful in analyzing the steady-state response of finite element models to harmonic forcing functions. In this context, a harmonic forcing function is described as $F(t) = F_0 \sin \omega_f t$, where F_0 is a constant force magnitude and ω_f is a constant circular frequency of the forcing function. Prior to applying the mode superposition method, a complete modal analysis must be performed to obtain the natural circular frequencies and normalized modal amplitude vectors (hence, the normalized modal matrix). Using the techniques of the previous section, the equations of motion for the forced case become

$$[I]\{\ddot{p}\} + [\omega^2]\{p\} = [A]^T\{F\} \tag{10.120}$$

Assuming that the structural model under consideration exhibits P total degrees of freedom, Equation 10.120 represents a set of P uncoupled, ordinary differential equations of the form

$$\ddot{p}_i + \omega_i^2 p_i = \sum_{j=1}^{P} A_j^{(i)} F_j(t) \qquad i = 1, P \tag{10.121}$$

Observing that the right-hand side is a known linear combination of harmonic forces (these are the so-called generalized forces), the solution to each of the equations is a summation of particular solutions corresponding to each of the harmonic force terms. By analogy with the procedure used for forced vibration of a single degree-of-freedom system in Section 10.2, the solutions of Equation 10.121 are given by

$$p_i(t) = \sum_{j=1}^{P} \frac{A_j^{(i)} F_{j0}}{\omega_i^2 - \omega_{jf}^2} \sin \omega_{jf} t \qquad i = 1, P \tag{10.122}$$

Hence, the generalized displacements $p_i(t)$ are represented by a combination of independent harmonic motions having frequencies corresponding to the forcing frequencies. Note that, if a forcing frequency is close in value to one of the natural frequencies, the denominator term becomes small and the forced response amplitude is large; hence, there are many possibilities for a resonant condition.

The mode superposition method provides mathematical convenience in obtaining the forced response, because the equations of motion become uncoupled and solution is straightforward. However, Equation 10.122 gives the displacement response of generalized displacements rather than actual nodal displacements, owing to the transformation described by Equation 10.112. As the modal matrix is known, conversion of the generalized displacements to actual displacements requires only multiplication by the normalized modal matrix.

EXAMPLE 10.8

Again consider the 3 degrees-of-freedom system of Example 10.3 and determine the steady state response when a downward force $F = F_0 \sin \omega_f t$ is applied to mass 2.

■ Solution

For the given conditions, the applied nodal force vector is

$$\{F(t)\} = \begin{Bmatrix} 0 \\ F_0 \sin \omega_f t \\ 0 \end{Bmatrix}$$

and the generalized forces are

$$[A]^T\{F\} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.2956 & 0.4209 & 0.6064 \\ 0.6575 & 0.5618 & -0.3550 \\ 0.6930 & -0.7124 & 0.0782 \end{bmatrix}\begin{Bmatrix} 0 \\ F_0 \sin \omega_f t \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.4209 \\ 0.5618 \\ -0.7124 \end{Bmatrix}\frac{F_0 \sin \omega_f t}{\sqrt{m}}$$

The equations of motion for the generalized coordinates are then

$$\ddot{p}_1 + \omega_1^2 p_1 = \frac{0.4209 F_0 \sin \omega_f t}{\sqrt{m}}$$

$$\ddot{p}_2 + \omega_2^2 p_2 = \frac{0.5618 F_0 \sin \omega_f t}{\sqrt{m}}$$

$$\ddot{p}_3 + \omega_3^2 p_3 = \frac{-0.7124 F_0 \sin \omega_f t}{\sqrt{m}}$$

for which the solutions are

$$p_1(t) = \frac{0.4209 F_0 \sin \omega_f t}{\left(\omega_1^2 - \omega_f^2\right)\sqrt{m}}$$

$$p_2(t) = \frac{0.5618 F_0 \sin \omega_f t}{\left(\omega_2^2 - \omega_f^2\right)\sqrt{m}}$$

$$p_3(t) = \frac{-0.7124 F_0 \sin \omega_f t}{\left(\omega_3^2 - \omega_f^2\right)\sqrt{m}}$$

The actual displacements, $x(t) = q(t)$ in this case, are obtained by application of Equation 10.112:

$$\{x\} = [A]\{p\} = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.2956 & 0.6575 & 0.6930 \\ 0.4209 & 0.5618 & -0.7124 \\ 0.6064 & -0.3550 & 0.0782 \end{bmatrix}\begin{Bmatrix} \dfrac{0.4209}{\omega_1^2 - \omega_f^2} \\ \dfrac{0.5618}{\omega_2^2 - \omega_f^2} \\ \dfrac{-0.7124}{\omega_3^2 - \omega_f^2} \end{Bmatrix}\frac{F_0 \sin \omega_f t}{\sqrt{m}}$$

Expanding, the steady-state displacements are given by

$$x_1(t) = \left(\frac{0.1244}{\omega_1^2 - \omega_f^2} + \frac{0.3694}{\omega_2^2 - \omega_f^2} + \frac{-0.4937}{\omega_3^2 - \omega_f^2}\right)\frac{F_0 \sin \omega_f t}{m}$$

$$x_2(t) = \left(\frac{0.1772}{\omega_1^2 - \omega_f^2} + \frac{0.3156}{\omega_2^2 - \omega_f^2} + \frac{0.5075}{\omega_3^2 - \omega_f^2}\right)\frac{F_0 \sin \omega_f t}{m}$$

$$x_3(t) = \left(\frac{0.2552}{\omega_1^2 - \omega_f^2} + \frac{-0.1994}{\omega_2^2 - \omega_f^2} + \frac{-0.0557}{\omega_3^2 - \omega_f^2}\right)\frac{F_0 \sin \omega_f t}{m}$$

A few observations need to be made regarding the displacements calculated in this example:

1. The displacement of each mass is a sinusoidal oscillation about the equilibrium position, and the circular frequency of the oscillation is the same as the frequency of the forcing function.
2. The characterstics of the principal modes of vibration are reflected in the solutions, owing to the effects of the natural circular frequencies and modal amplitude vectors in determining the forced oscillation amplitudes.
3. The displacement solutions represent only the forced motion of each mass; in addition, free vibration may also exist in superposition with the forced response.
4. Energy dissipation mechanisms are not incorporated into the model.

The mode superposition method may seem quite complicated and, when attempting to obtain solutions by hand, the method is indeed tedious. However, the required computations are readily amenable to digital computer techniques and quite easily programmed. Additional ramifications of computer techniques for the method will be discussed in a following article.

10.9 ENERGY DISSIPATION: STRUCTURAL DAMPING

To this point, the dynamic analysis techniques dealt only with structural systems in which there is no mechanism for energy dissipation. As stated earlier, all real systems exhibit such dissipation and, unlike the simple models presented, do not oscillate forever, as predicted by the ideal model solutions. In structural systems, the phenomenon of energy dissipation is referred to as *damping*. Damping may take on many physical forms, including devices specifically designed for the purpose (passive and active damping devices), sliding friction, and the internal dissipation characteristics of materials subjected to cyclic loading. In this section, we begin with an idealized model of damping for the simple harmonic oscillator and extend the damping concept to full-scale structural models.

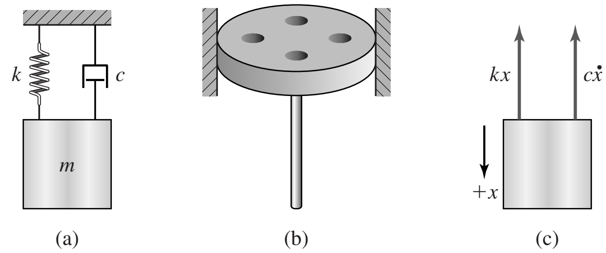

Figure 10.13
(a) A spring-mass system with damping. (b) The schematic representation of a dashpot piston. (c) A free-body diagram of a mass with the damping force included.

Figure 10.13a depicts a simple harmonic oscillator to which has been added a *dashpot*. A *dashpot* is a damping device that utilizes a piston moving through a viscous fluid to remove energy via shear stress in the fluid and associated heat generation. The piston typically has small holes to allow the fluid to pass through but is otherwise sealed on its periphery, as schematically depicted in Figure 10.13b. The force exerted by such a device is known to be directly proportional to the velocity of the piston as

$$f_d = -c\dot{x} \tag{10.123}$$

where f_d is the damping force, c is the damping coefficient of the device, and $\dot{x}$ is velocity of the mass assumed to be directly and rigidly connected to the piston of the damper. The dynamic free-body diagram of Figure 10.13c represents a situation at an arbitrary time with the system in motion. As in the undamped case considered earlier, we assume that displacement is measured from the equilibrium position. Under the conditions stated, the equation of motion of the mass is

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{10.124}$$

Owing to the form of Equation 10.124, the solution is assumed in exponential form as

$$x(t) = Ce^{st} \tag{10.125}$$

where C and s are constants to be determined. Substitution of the assumed solution yields

$$(ms^2 + cs + k)Ce^{st} = 0 \tag{10.126}$$

As we seek nontrivial solutions valid for all values of time, we conclude that

$$ms^2 + cs + k = 0 \tag{10.127}$$

must hold if we are to obtain a general solution. Equation 10.127 is the characteristic equation (also the frequency equation) for the damped single degree-of-freedom system. From analyses of undamped vibration, we know that the natural frequency given by $\omega^2 = k/m$ is an important property of the system, so we modify the characteristic equation to

$$s^2 + \frac{c}{m}s + \omega^2 = 0 \tag{10.128}$$

Solving Equation 10.128 by the quadratic formula yields two roots, as expected, given by

$$s_1 = \frac{1}{2}\left[\left(\frac{c}{m}\right)^2 - \sqrt{\left(\frac{c}{m}\right)^2 - 4\omega^2}\right] \tag{10.129a}$$

$$s_2 = \frac{1}{2}\left[\left(\frac{c}{m}\right)^2 + \sqrt{\left(\frac{c}{m}\right)^2 - 4\omega^2}\right] \tag{10.129b}$$

The most important characteristic of the roots is the value of $(c/m)^2 - 4\omega^2$, and there are three cases of importance:

1. If $(c/m)^2 - 4\omega^2 > 0$, the roots are real, distinct, and negative; and the displacement response is the sum of decaying exponentials.
2. If $(c/m)^2 - 4\omega^2 = 0$, we have a case of repeated roots; for this situation, the displacement is also shown to be a decaying exponential. It is convenient to define this as a critical case and let the value of the damping coefficient c correspond to the so-called *critical damping coefficient*. Hence, $c_c^2 = 4\omega^2 m^2$ or $c_c = 2m\omega$.
3. If $(c/m)^2 - 4\omega^2 < 0$, the roots of the characteristic equation are imaginary; this case can be shown [2] to represent decaying sinusoidal oscillations.

Regardless of the amount of damping present, the free-vibration response, as shown by the preceding analysis, is an exponentially decaying function in time. This gives more credence to our previous discussion of harmonic response, in which we ignored the free vibrations. In general, a system response is defined primarily by the applied forcing functions, as the natural (free, principal) vibrations die out with damping. The response of a damped spring-mass system corresponding to each of the three cases of damping is depicted in Figure 10.14.

We now define the *damping ratio* as $\zeta = c/2m\omega$ and note that, if $\zeta > 1$, we have what is known as *overdamped* motion; if $\zeta = 1$, the motion is said to be *critically damped;* and if $\zeta < 1$, the motion is *underdamped*. As most structural systems are underdamped, we focus on the case of $\zeta < 1$. For this situation, it is readily shown [2] that the response of a damped harmonic oscillator is described by

$$x(t) = e^{-\zeta\omega t}(A \sin \omega_d t + B \cos \omega_d t) \tag{10.130}$$

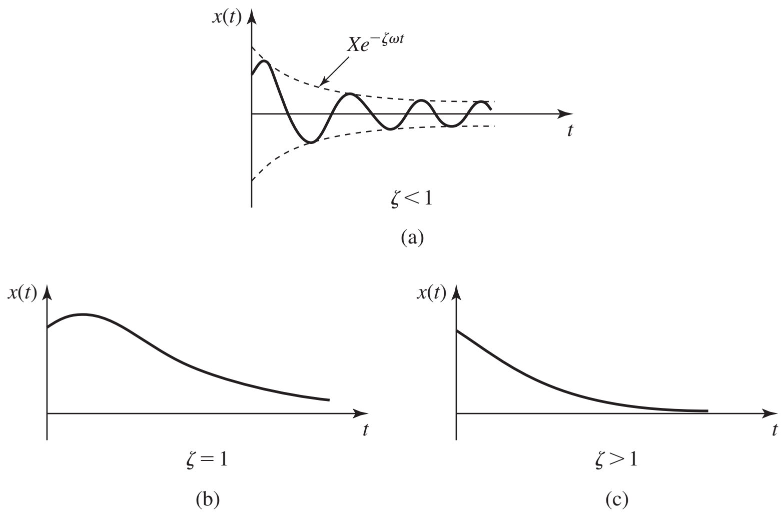

Figure 10.14 Characteristic damped motions: (a) Underdamped. (b) Critically damped. (c) Overdamped.

where ω_d is the *damped natural circular frequency,* given by

$$\omega_d^2 = (1 - \zeta^2)\frac{k}{m} \tag{10.131}$$

and the coefficients are determined by the initial conditions.

While we demonstrate the effect of damping via the simple harmonic oscillator, several points can be made that are applicable to any structural system:

1. The natural frequencies of vibration of a system are reduced by the effect of damping, per Equation 10.131.
2. The free vibrations decay exponentially to zero because of the effects of damping.
3. In light of point 2, in the case of forced vibration, the steady-state solution is driven only by the forcing functions.
4. Damping is assumed to be linearly proportional to nodal velocities.

10.9.1 General Structural Damping

An elastic structure subjected to dynamic loading does not, in general, have specific damping elements attached. Instead, the energy dissipation characteristics of the structure are inherent to its mechanical properties. How does, for example, a cantilevered beam, when "tweaked" at one end, finally stop vibrating? (If the reader has a flexible ruler at hand, many experiments can be performed to exhibit the change in fundamental frequency as a function of beam length as well as the

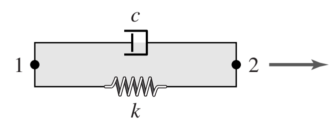

Figure 10.15 A model of a bar element with damping.

decay of the motion.) The answer to the damping question is complex. For example, structures are subjected to the atmosphere, so that air resistance is a factor. Air resistance is, in general, proportional to velocity squared, so this effect is nonlinear. Fortunately, air resistance in most cases is negligible. On the other hand, the internal friction of a material is not negligible and must be considered.

If we incorporate the concepts of damping as applied to the simple harmonic oscillator, the equations of motion of a finite element model of a structure become

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{F(t)\} \tag{10.132}$$

where $[C]$ is the system viscous damping matrix assembled by the usual rules. For example, a bar element with damping is mathematically modeled as a linear spring and a dashpot connected in parallel to the element nodes as in Figure 10.15. The element damping matrix is

$$\left[c^{(e)}\right] = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \tag{10.133}$$

and the element equations of motion are

$$\left[m^{(e)}\right]\{\ddot{u}\} + \left[c^{(e)}\right]\{\dot{u}\} + \left[k^{(e)}\right]\{u\} = \left\{f^{(e)}\right\} \tag{10.134}$$

The element damping matrix is symmetric and singular, and the individual terms are assigned to the global damping matrix in the same manner as the mass and stiffness matrices. Assembly of the global equations of motion for a finite element model of a damped structure is simple. Determination of the effective viscous damping coefficients for structural elements is not so simple.

Damping due to internal friction is known as *structural damping,* and experiments on many different elastic materials have shown that the energy loss per motion cycle in structural damping is proportional to the material stiffness and the square of displacement amplitude [2]. That is,

$$\Delta U_{\text{cycle}} = \lambda k X^2 \tag{10.135}$$

where λ is a dimensionless structural damping coefficient, k is the material stiffness, and X is the displacement amplitude. By equating the energy loss per cycle to the energy loss per cycle in viscous damping, an equivalent viscous damping coefficient is obtained:

$$c_{\text{eq}} = \frac{\lambda k}{\omega} \tag{10.136}$$

where ω is circular frequency of oscillation. That the equivalent damping coefficient depends on frequency is somewhat troublesome, since the implication is that different coefficients are required for different frequencies. If we consider a single degree-of-freedom system for which $\omega = \sqrt{k/m}$, the equivalent damping coefficient given by Equation 10.136 becomes

$$c_{\text{eq}} = \frac{\lambda k}{\omega} = \lambda \frac{k}{\sqrt{k/m}} = \lambda \sqrt{km} \tag{10.137}$$

indicating that the damping coefficient is proportional, at least in a general sense, to both stiffness and mass. We return to this observation shortly.

Next we consider the application of the transformation using the normalized matrix as described in Section 10.7. Applying the transformation to Equation 10.132 results in

$$\{\ddot{p}\} + [A]^T[C][A]\{\dot{p}\} + [\omega^2]\{p\} = [A]^T\{F(t)\} \tag{10.138}$$

The transformed damping matrix

$$[C'] = [A]^T[C][A] \tag{10.139}$$

is easily shown to be a symmetric matrix, but the matrix is *not necessarily diagonal.* The transformation does not necessarily result in decoupling the equations of motion, and the simplification of the mode superposition method is not necessarily available. If, however, the damping matrix is such that

$$[C] = \alpha[M] + \beta[K] \tag{10.140}$$

where α and β are constants, then

$$[C'] = \alpha[A]^T[M][A] + \beta[A]^T[K][A] = \alpha[I] + \beta[\omega^2] \tag{10.141}$$

is a diagonal matrix and the differential equations of motion are decoupled. Note that the assertion of Equation 10.140 leads directly to the diagonalization of the damping matrix as given by Equation 10.141. Hence, Equation 10.138 becomes

$$\{\ddot{p}\} + (\alpha + \beta[\omega^2])\{\dot{p}\} + [\omega^2]\{p\} = [A]^T\{F(t)\} \tag{10.142}$$

As the differential equations represented by Equation 10.142 are decoupled, let us now examine the solution of one such equation

$$\ddot{p}_i + \left(\alpha + \beta\omega_i^2\right)\dot{p}_i + \omega_i^2 p_i = \sum_{j=1}^{P} A_j^{(i)} F_j(t) \tag{10.143}$$

where P is the total number of degrees of freedom. Without loss of generality and for convenience of illustration, we consider Equation 10.143 for only one of the terms on the right-hand side, assumed to be a harmonic force such that

$$\ddot{p}_i + \left(\alpha + \beta\omega_i^2\right)\dot{p}_i + \omega_i^2 p_i = F_0 \sin \omega_f t \tag{10.144}$$

and assume that the solution is

$$p_i(t) = X_i \sin \omega_f t + Y_i \cos \omega_f t \tag{10.145}$$

Substitution of the assumed solution into the governing equation yields

$$-X_i\omega_f^2 \sin\omega_f t - Y_i\omega_f^2 \cos\omega_f t + \left(\alpha + \beta\omega_i^2\right)\omega_f(X_i\cos\omega_f t - Y_i \sin\omega_f t) + \omega_i^2 X_i \sin\omega_f t + \omega_i^2 Y_i \cos\omega_f t = F_0 \sin\omega_f t \tag{10.146}$$

Equating coefficients of sine and cosine terms yields the algebraic equations

$$\begin{bmatrix} \omega_i^2 - \omega_f^2 & -\omega_f\left(\alpha + \beta\omega_i^2\right) \\ \omega_f\left(\alpha + \beta\omega_i^2\right) & \omega_i^2 - \omega_f^2 \end{bmatrix} \begin{Bmatrix} X_i \\ Y_i \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \tag{10.147}$$

for determination of the forced amplitudes X_i and Y_i. The solutions are

$$\begin{aligned} X_i &= \frac{F_0\left(\omega_i^2 - \omega_f^2\right)}{\left(\omega_i^2 - \omega_f^2\right)^2 + \omega_f^2\left(\alpha + \beta\omega_i^2\right)^2} \\ Y_i &= \frac{-F_0\omega_f\left(\alpha + \beta\omega_i^2\right)}{\left(\omega_i^2 - \omega_f^2\right)^2 + \omega_f^2\left(\alpha + \beta\omega_i^2\right)^2} \end{aligned} \tag{10.148}$$

To examine the character of the solution represented by Equation 10.145, we convert the solution to the form

$$p_i(t) = Z_i \sin(\omega_f t + \phi_i) \tag{10.149}$$

with

$$Z_i = \sqrt{X_i^2 + Y_i^2} \quad \text{and} \quad \phi_i = \tan^{-1}\frac{Y_i}{X_i}$$

to obtain

$$p_i(t) = \frac{F_0}{\sqrt{\left(\omega_i^2 - \omega_f^2\right)^2 + \omega_f^2\left(\alpha + \beta\omega_i^2\right)^2}} \sin(\omega_f t + \phi_i) \tag{10.150}$$

$$\phi_i = \tan^{-1}\left(\frac{-\omega_f^2\left(\alpha + \beta\omega_i^2\right)}{\omega_i^2 - \omega_f^2}\right) \tag{10.151}$$

Again, the mathematics required to obtain these solutions are algebraically tedious; however, Equations 10.150 and 10.151 are perfectly general, in that the equations give the solution for every equation in 10.142, provided the applied nodal forces are harmonic. Such solutions are easily generated via digital computer software. The actual displacements are then obtained by application of Equation 10.112, as in the case of undamped systems.

The equivalent viscous damping described in Equation 10.140 is known as *Rayleigh damping* [6] and used very often in structural analysis. It can be shown, by comparison to a damped single degree-of-freedom system that

$$\alpha + \beta\omega_i^2 = 2\omega_i\zeta_i \tag{10.152}$$

where ζ_i is the damping ratio corresponding to the ith mode of vibration, that is,

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \tag{10.153}$$

represents the degree of damping for the ith mode. Equation 10.153 provides a means of estimating α and β if realistic estimates of the degree of damping for two modes are known. The realistic estimates are most generally obtained experimentally or may be applied by rule of thumb. The following example illustrates the computations and the effect on other modes.

EXAMPLE 10.9

Experiments on a prototype structure indicate that the effective viscous damping ratio is $\zeta = 0.03$ (3 percent) when the oscillation frequency is $\omega = 5$ rad/sec and $\zeta = 0.1$ (10 percent) for frequency $\omega = 15$ rad/sec. Determine the Rayleigh damping factors α and β for these known conditions.

■ **Solution**

Applying Equation 10.153 to each of the known conditions yields

$$0.03 = \frac{\alpha}{2(5)} + \frac{5\beta}{2}$$
$$0.1 = \frac{\alpha}{2(15)} + \frac{15\beta}{2}$$

Simultaneous solution provides the Rayleigh coefficients as

$$\alpha = -0.0375$$
$$\beta = 0.0135$$

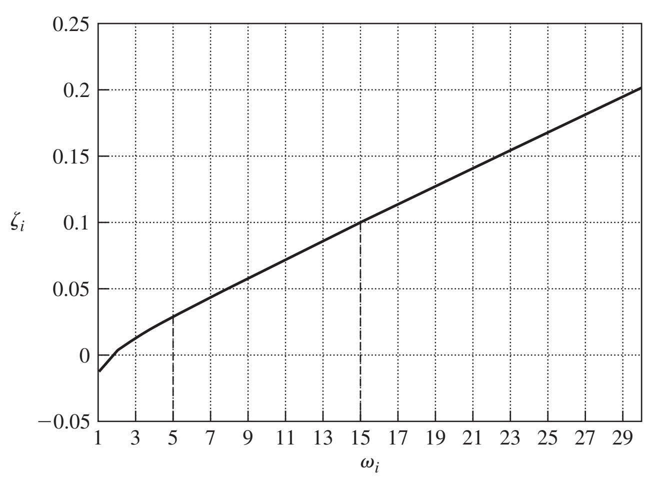

Figure 10.16 Equivalent damping factor versus frequency for Example 10.9.

If we were to apply the equivalent damping given by these values to the entire frequency spectrum of a structure, the effective damping ratio for any mode would be given by

$$\zeta_i = \frac{-0.0375 + 0.0135\omega_i^2}{2\omega_i}$$

If the values of α and β are applied to a multiple degrees-of-freedom system, the damping ratio for each frequency is different. To illustrate the variation, Figure 10.16 depicts the modal damping ratio as a function of frequency. The plot shows that, of course, the ratios for the specified frequencies are exact and the damping ratios vary significantly for other frequencies.

Rayleigh damping as just described is not the only approach to structural damping used in finite element analysis. Finite element software packages also include options for specifying damping as a material-dependent property, as opposed to a property of the structure, as well as defining specific damping elements (finite elements) that may be added at any geometric location in the structure. The last capability allows the finite element analyst to examine the effects of energy dissipation elements as applied to specific locations.

10.10 TRANSIENT DYNAMIC RESPONSE

In Chapter 7, finite difference methods for direct numerical integration of finite element models of heat transfer problems are introduced. In those applications, we deal with a scalar field variable, temperature, and first-order governing equations. Therefore, we need only to develop finite difference approximations to first derivatives. For structural dynamic systems, we have a set of second-order differential equations

$$[M]\{\ddot{\delta}\} + [C]\{\dot{\delta}\} + [K]\{\delta\} = \{F(t)\} \tag{10.154}$$

representing the assembled finite element model of a structure subjected to general (nonharmonic) forcing functions. In applying finite difference methods to Equation 10.154, we assume that the state of the system is known at time t and we wish to compute the displacements at time $t + \Delta t$; that is, we wish to solve

$$[M]\{\ddot{\delta}(t + \Delta t)\} + [C]\{\dot{\delta}(t + \Delta t)\} + [K]\{\delta(t + \Delta t)\} = \{F(t + \Delta t)\} \tag{10.155}$$

for $\{\delta(t + \Delta t)\}$.

Many finite difference techniques exist for solving the system of equations represented by Equation 10.155. Here, we describe Newmark's method [7] also referred to as the *constant acceleration method.* In the Newmark method, it is assumed that the acceleration during an integration time step Δt is constant and an average value. For constant acceleration, we can write the kinematic relations

$$\delta(t + \Delta t) = \delta(t) + \dot{\delta}(t)\Delta t + \ddot{\delta}_{av}\frac{\Delta t^2}{2} \tag{10.156}$$

$$\dot{\delta}(t + \Delta t) = \dot{\delta}(t) + \ddot{\delta}_{av}\Delta t \tag{10.157}$$

The constant, average acceleration is

$$\ddot{\delta}_{\text{av}} = \frac{\ddot{\delta}(t+\Delta t) + \ddot{\delta}(t)}{2} \tag{10.158}$$

Combining Equations 10.156 and 10.158 yields

$$\delta(t+\Delta t) = \delta(t) + \dot{\delta}(t)\Delta t + [\ddot{\delta}(t+\Delta t) + \ddot{\delta}(t)]\frac{\Delta t^2}{4} \tag{10.159}$$

which is solved for the acceleration at $t + \Delta t$ to obtain

$$\ddot{\delta}(t+\Delta t) = \frac{4}{\Delta t^2}[\delta(t+\Delta t) - \delta(t)] - \frac{4}{\Delta t}\dot{\delta}(t) - \ddot{\delta}(t) \tag{10.160}$$

If we also substitute Equations 10.158 and 10.160 into Equation 10.157, we find the velocity at time $t + \Delta t$ to be given by

$$\dot{\delta}(t+\Delta t) = \frac{2}{\Delta t}[\delta(t+\Delta t) - \delta(t)] - \dot{\delta}(t) \tag{10.161}$$

Equations 10.160 and 10.161 express acceleration and velocity at $t + \Delta t$ in terms of known conditions at the previous time step and the displacement at $t + \Delta t$. If these relations are substituted into Equation 10.155, we obtain, after a bit of algebraic manipulation,

$$\begin{aligned}\frac{4}{\Delta t^2}&[M]\{\delta(t+\Delta t)\} + \frac{2}{\Delta t}[C]\{\delta(t+\Delta t)\} + [K]\{\delta(t+\Delta t)\}\\ &= \{F(t+\Delta t)\} + [M]\left(\{\ddot{\delta}(t)\} + \frac{4}{\Delta t}\{\dot{\delta}(t)\} + \frac{4}{\Delta t^2}\{\delta(t)\}\right)\\ &\quad + [C]\left(\{\dot{\delta}(t)\} + \frac{2}{\Delta t}\{\delta(t)\}\right)\end{aligned} \tag{10.162}$$

Equation 10.162 is the recurrence relation for the Newmark method. While the relation may look complicated, it must be realized that the mass, damping, and stiffness matrices are known, so the equations are just an algebraic system in the unknown displacements at time $t + \Delta t$. The right-hand side of the system is known in terms of the solution at the previous time step and the applied forces. Equation 10.162 is often written symbolically as

$$[\bar{K}]\{\delta(t+\Delta t)\} = \{F_{\text{eff}}(t+\Delta t)\} \tag{10.163}$$

with

$$[\bar{K}] = \frac{4}{\Delta t^2}[M] + \frac{2}{\Delta t}[C] + [K] \tag{10.164}$$

$$\begin{aligned}\{F_{\text{eff}}(t+\Delta t)\} = \{&F(t+\Delta t)\}\\ &+ [M]\left(\{\ddot{\delta}(t)\} + \frac{4}{\Delta t}\{\dot{\delta}(t)\} + \frac{4}{\Delta t^2}\{\delta(t)\}\right)\\ &+ [C]\left(\{\dot{\delta}(t)\} + \frac{2}{\Delta t}\{\delta(t)\}\right)\end{aligned} \tag{10.165}$$

The system of algebraic equations represented by Equation 10.163 can be solved at each time step for the unknown displacements. For a constant time step Δt, matrix $[\bar{K}]$ is constant and need be computed only once. The right-hand side $\{F_{\text{eff}}(t+\Delta t)\}$ must, of course, be updated at each time step. At each time step, the system of algebraic equations must be solved to obtain displacements. For this reason, the procedure is known as an implicit method. By back substitution through the appropriate relations, velocities and accelerations can also be obtained.

The Newmark method is known to be *unconditionally stable* [8]. While the details are beyond the scope of this text, stability (more to the point, instability) of a finite difference technique means that, under certain conditions, the computed displacements may grow without bound as the solution procedure "marches" in time. Several finite difference methods are known to be *conditionally stable,* meaning that accurate results are obtained only if the time step Δt is less than a prescribed critical value. This is not the case with the Newmark method. This does not mean, however, that the results are independent of the selected time step. Accuracy of any finite difference technique improves as the time step is reduced, and this phenomenon is a convergence concern similar to mesh refinement in a finite element model. For dynamic response of a finite element model, we must be concerned with not only the convergence related to the finite element mesh but also the time step convergence of the finite difference method selected. As discussed in a following section, finite element software for the transient dynamic response requires the user to specify "load steps," which represent the change in loading as a function of time. The software then solves the finite element equations as if the problem is one of static equilibrium at the specified loading condition. It is very important to note that the system equations represented by Equation 10.163 are based on the finite element model, even though the solution procedure is that of the finite difference technique in time.

10.11 BAR ELEMENT MASS MATRIX IN TWO-DIMENSIONAL TRUSS STRUCTURES

The bar-element-consistent mass matrix defined in Equation 10.58 is valid only for axial vibrations. When bar elements are used in modeling two- and three-dimensional truss structures, additional considerations are required, and the mass matrix modified accordingly. When a truss undergoes deflection, either statically or dynamically, individual elements experience both axial and transverse displacement resulting from overall structural displacement and element interconnections at nodes. In Chapter 3, transverse displacement of elements was ignored in development of the element stiffness matrix as there is no transverse stiffness owing to the assumption of pin connections, hence free rotation. However, in the dynamic case, transverse motion introduces additional kinetic energy, which must be taken into account.

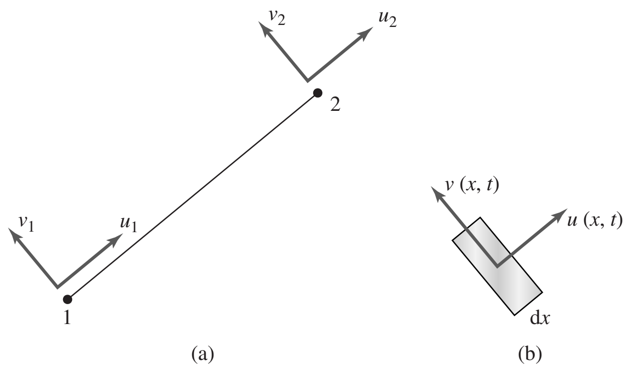

Figure 10.17 A bar element in two-dimensional motion: (a) Nodal displacements. (b) Differential element.

Consider the differential volume of a bar element undergoing both axial and transverse displacement, as shown in Figure 10.17. We assume a dynamic situation such that both displacement components vary with position and time. The kinetic energy of the differential volume is

$$\mathrm{d}T = \frac{1}{2}\rho A\,\mathrm{d}x\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right] = \frac{1}{2}\rho A\,\mathrm{d}x(\dot{u}^2 + \dot{v}^2) \tag{10.166}$$

and the total kinetic energy of the bar becomes

$$T = \frac{1}{2}\rho A\int_0^L \dot{u}^2\,\mathrm{d}x + \frac{1}{2}\rho A\int_0^L \dot{v}^2\,\mathrm{d}x \tag{10.167}$$

Observing that the transverse displacement can be expressed in terms of the transverse displacements of the element nodes, using the same interpolation functions as for axial displacement, we have

$$\begin{aligned} u(x, t) &= N_1(x)u_1(t) + N_2(x)u_2(t) \\ v(x, t) &= N_1(x)v_1(t) + N_2(x)v_2(t) \end{aligned} \tag{10.168}$$

Using matrix notation, the velocities are written as

$$\begin{aligned} \dot{u}(x, t) &= [N_1 \quad N_2]\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \\ \dot{v}(x, t) &= [N_1 \quad N_2]\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} \end{aligned} \tag{10.169}$$

and element kinetic energy becomes

$$T = \frac{1}{2}\rho A\{\dot{u}\}^T\int_0^L [N]^T[N]\,\mathrm{d}x\{\dot{u}\} + \frac{1}{2}\rho A\{\dot{v}\}^T\int_0^L [N]^T[N]\,\mathrm{d}x\{\dot{v}\} \tag{10.170}$$

Expressing the nodal velocities as

$$\{\dot{\delta}\} = \begin{Bmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{u}_2 \\ \dot{v}_2 \end{Bmatrix} \tag{10.171}$$

the kinetic energy expression can be rewritten in the form

$$\begin{aligned} T &= \frac{1}{2}\{\dot{\delta}\}^T \left[m_2^{(e)}\right]\{\dot{\delta}\} \\ &= \frac{1}{2}\{\dot{\delta}\}^T \rho A \int_0^L \begin{bmatrix} N_1^2 & 0 & N_1N_2 & 0 \\ 0 & N_1^2 & 0 & N_1N_2 \\ N_1N_2 & 0 & N_2^2 & 0 \\ 0 & N_1N_2 & 0 & N_2^2 \end{bmatrix} \mathrm{d}x\{\dot{\delta}\} \end{aligned} \tag{10.172}$$

From Equation 10.172, the mass matrix of the bar element in two dimensions is identified as

$$\left[m_2^{(e)}\right] = \rho A \int_0^L \begin{bmatrix} N_1^2 & 0 & N_1N_2 & 0 \\ 0 & N_1^2 & 0 & N_1N_2 \\ N_1N_2 & 0 & N_2^2 & 0 \\ 0 & N_1N_2 & 0 & N_2^2 \end{bmatrix} \mathrm{d}x = \frac{\rho AL}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{10.173}$$

The mass matrix defined by Equation 10.173 is described in the element (local) coordinate system, since the axial and transverse directions are defined in terms of the axis of the element. How, then, is this mass matrix transformed to the global coordinate system of a structure? Recall that, in Chapter 3, the element axial displacements are expressed in terms of global displacements via a rotation transformation of the element x axis. To reiterate, the transverse displacements were not considered, as no stiffness is associated with the transverse motion. Now, however, the transverse displacements must be included in the transformation to global coordinates because of the associated mass and kinetic energy.

Figure 10.18 depicts a single node of a bar element oriented at angle θ relative to the X axis of a global coordinate system. Nodal displacements in the element frame are u_2, v_2 and corresponding global displacements are U_3, U_4, respectively. As the displacement in the two coordinate systems must be the same, we have

$$\begin{aligned} u_2 &= U_3 \cos\theta + U_4 \sin\theta \\ v_2 &= -U_3 \sin\theta + U_4 \cos\theta \end{aligned} \tag{10.174}$$

or

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} \tag{10.175}$$

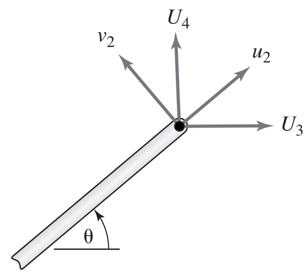

Figure 10.18 The relation of element and global displacements at a single node.

As the same relation holds at the other element node, the complete transformation is

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = [R]\{U\} \tag{10.176}$$

Since the nodal velocities are related by the same transformation, substitution into the kinetic energy expression shows that the mass matrix in the global coordinate system is

$$\left[M_2^{(e)}\right] = [R]^T\left[m_2^{(e)}\right][R] \tag{10.177}$$

where we again use the subscript to indicate that the mass matrix is applicable to two-dimensional structures.

If the matrix multiplications indicated in Equation 10.177 are performed for an *arbitrary* angle, the resulting global mass matrix for a bar element is found to be

$$\left[M_2^{(e)}\right] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{10.178}$$

and the result is exactly the same as the mass matrix in the element coordinate system *regardless* of element orientation in the global system. This phenomenon should come as no surprise. Mass is an absolute scalar property and therefore independent of coordinate system. A similar development leads to the same conclusion when a bar element is used in modeling three-dimensional truss structures.

The complication described for including the additional transverse inertia effects of the bar element are also applicable to the one-dimensional beam (flexure) element. The mass matrix for the beam element given by Equation 10.78 is applicable only in a one-dimensional model. If the flexure element is used in modeling two- or three-dimensional frame structures, additional consideration must be given to formulation of the element mass matrix owing to axial inertia effects. For beam elements, most finite software packages include axial effects (i.e., the beam element is a combination of the bar element and the two-dimensional flexure element) and all appropriate inertia effects are included in formulation of the consistent mass matrix.

EXAMPLE 10.10

As a complete example of modal analysis, we return to the truss structure of Section 3.7, repeated here as Figure 10.19. Note that, for the current example, the static loads applied in the earlier example have been removed. As we are interested here in the free-vibration response of the structure, the static loads are of no consequence in the dynamic analysis. With the additional specification that material density is $\rho = 2.6(10)^{-4}$ lb-s²/in.⁴, we solve the eigenvalue problem to determine the natural circular frequencies and modal amplitude vectors for free vibration of the structure.

As the global stiffness matrix has already been assembled, the procedure is not repeated here. We must, however, assemble the global mass matrix using the element numbers and global node numbers as shown. The element and global mass matrices for the bar element in two dimensions are given by Equation 10.178 as

$$\left[m^{(e)}\right] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

As elements 1, 3, 4, 5, 7, and 8 have the same length, area, and density, we have

$$\left[M^{(1)}\right] = \left[M^{(3)}\right] = \left[M^{(4)}\right] = \left[M^{(5)}\right] = \left[M^{(7)}\right] = \left[M^{(8)}\right]$$

$$= \frac{(2.6)(10)^{-4}(1.5)(40)}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5.2 & 0 & 2.6 & 0 \\ 0 & 5.2 & 0 & 2.6 \\ 2.6 & 0 & 5.2 & 0 \\ 0 & 2.6 & 0 & 5.2 \end{bmatrix}(10)^{-3} \text{ lb-s}^2/\text{in.}$$

while for elements 2 and 6

$$\left[M^{(2)}\right] = \left[M^{(6)}\right] = \frac{2.6(10)^{-4}(1.5)(40\sqrt{2})}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7.36 & 0 & 3.68 & 0 \\ 0 & 7.36 & 0 & 3.68 \\ 3.68 & 0 & 7.36 & 0 \\ 0 & 3.68 & 0 & 7.36 \end{bmatrix}(10)^{-3} \text{ lb-s}^2/\text{in.}$$

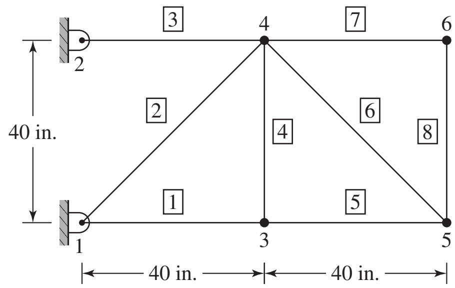

Figure 10.19 Eight-element truss of Example 10.10.

The element-to-global displacement relations are as given in Chapter 3. Using the direct assembly procedure, the global mass matrix is

$$[M] = \begin{bmatrix} 12.56 & 0 & 0 & 0 & 2.6 & 0 & 3.68 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12.56 & 0 & 0 & 0 & 2.6 & 0 & 3.68 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.2 & 0 & 0 & 0 & 2.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.2 & 0 & 0 & 0 & 2.6 & 0 & 0 & 0 & 0 \\ 2.6 & 0 & 0 & 0 & 7.8 & 0 & 2.6 & 0 & 2.6 & 0 & 0 & 0 \\ 0 & 2.6 & 0 & 0 & 0 & 7.8 & 0 & 2.6 & 0 & 2.6 & 0 & 0 \\ 3.68 & 0 & 2.6 & 0 & 2.6 & 0 & 22.52 & 0 & 3.68 & 0 & 2.6 & 0 \\ 0 & 3.68 & 0 & 2.6 & 0 & 2.6 & 0 & 22.52 & 0 & 3.68 & 0 & 2.6 \\ 0 & 0 & 0 & 0 & 2.6 & 0 & 3.68 & 0 & 17.76 & 0 & 2.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.6 & 0 & 3.68 & 0 & 17.76 & 0 & 2.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.6 & 0 & 2.6 & 0 & 10.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.6 & 0 & 2.6 & 0 & 10.4 \end{bmatrix} (10)^{-3}$$

Applying the constraint conditions $U_1 = U_2 = U_3 = U_4 = 0$, the mass matrix for the active degrees of freedom becomes

$$[M_a] = \begin{bmatrix} 7.8 & 0 & 2.6 & 0 & 2.6 & 0 & 0 & 0 \\ 0 & 7.8 & 0 & 2.6 & 0 & 2.6 & 0 & 0 \\ 2.6 & 0 & 22.52 & 0 & 3.68 & 0 & 2.6 & 0 \\ 0 & 2.6 & 0 & 22.52 & 0 & 3.68 & 0 & 2.6 \\ 2.6 & 0 & 3.68 & 0 & 17.76 & 0 & 2.6 & 0 \\ 0 & 2.6 & 0 & 3.68 & 0 & 17.76 & 0 & 2.6 \\ 0 & 0 & 2.6 & 0 & 2.6 & 0 & 10.4 & 0 \\ 0 & 0 & 0 & 2.6 & 0 & 2.6 & 0 & 10.4 \end{bmatrix} (10)^{-3} \text{ lb-s}^2\text{/in.}$$

Extracting the data from Section 3.7, the stiffness matrix for the active degrees of freedom is

$$[K_a] = \begin{bmatrix} 7.5 & 0 & 0 & 0 & -3.75 & 0 & 0 & 0 \\ 0 & 3.75 & 0 & -3.75 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10.15 & 0 & -1.325 & 1.325 & -3.75 & 0 \\ 0 & -3.75 & 0 & 6.4 & 1.325 & -1.325 & 0 & 0 \\ -3.75 & 0 & -1.325 & 1.325 & 5.075 & -1.325 & 0 & 0 \\ 0 & 0 & 1.325 & -1.325 & -1.325 & 5.075 & 0 & -3.75 \\ 0 & 0 & -3.75 & 0 & 0 & 0 & 3.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.75 & 0 & 3.75 \end{bmatrix} 10^5 \text{ lb/in.}$$

The finite element model for the truss exhibits 8 degrees of freedom; hence, the characteristic determinant

$$|-\omega^2[M] + [K]| = 0$$

yields, theoretically, eight natural frequencies of oscillation and eight corresponding modal shapes (modal amplitude vectors). For this example, the natural modes were computed using the student edition of the ANSYS program [9], with the results shown in Table 10.1. The corresponding modal amplitude vectors (normalized to the mass matrix as discussed relative to orthogonality) are shown in Table 10.2.

The frequencies are observed to be quite large in magnitude. The fundamental frequency, about 122 cycles/sec is beyond the general comprehension of the human eye-brain interface (30 Hz is the accepted cutoff based on computer graphics research [10]). The high frequencies are not uncommon in such structures. The data used in this example correspond approximately to the material properties of aluminum; a light material with good stiffness relative to weight. Recalling the basic relation $\omega = \sqrt{k/m}$, high natural frequencies should be expected.

The mode shapes provide an indication of the geometric nature of the natural modes. As such, the numbers in Table 10.2 are not at all indicative of amplitude values; instead,

Table 10.1 Natural Modes

	Frequency	
Mode	**Rad/sec**	**Hz**
1	767.1	122.1
2	2082.3	331.4
3	2958.7	470.9
4	4504.8	716.9
5	6790.9	1080.8
6	7975.9	1269.4
7	8664.5	1379.0
8	8977.4	1428.8

Table 10.2 Modal Amplitude Vectors

	Mode							
Displacement	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
U_5	0.2605	2.194	1.213	−3.594	−1.445	−1.802	4.772	−4.368
U_6	2.207	−3.282	3.125	−2.1412	5.826	−0.934	1.058	0.727
U_7	−0.7754	0.7169	2.888	2.370	−0.142	−3.830	−2.174	−0.464
U_8	2.128	−2.686	1.957	−0.4322	−4.274	0.569	−0.341	0.483
U_9	0.5156	3.855	1.706	−3.934	−0.055	1.981	−2.781	3.956
U_{10}	4.118	2.556	−1.459	1.133	0.908	1.629	−3.319	−4.407
U_{11}	−0.7894	0.9712	4.183	4.917	0.737	6.077	4.392	−1.205
U_{12}	4.213	2.901	−1.888	2.818	0.604	−3.400	4.828	5.344

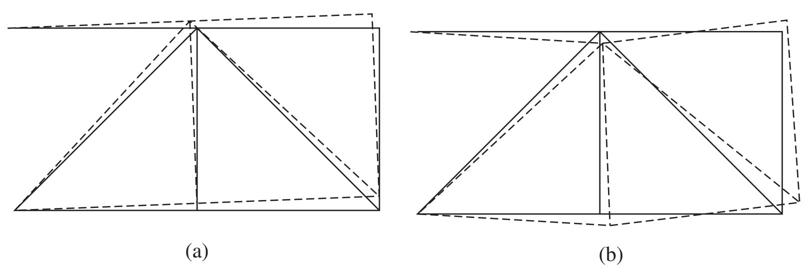

Figure 10.20
(a) Fundamental mode shape of the truss in Example 10.10.
(b) Second mode shape of the truss.

these are *relative* values of the motion of each node. It is more insightful to examine plots of the *mode shapes;* that is, plots of the structure depicting the shape of the structure if it did indeed oscillate in one of its natural modes. To this end, we present the mode shape corresponding to mode 1 in Figure 10.20a. Note that, in this fundamental mode, the truss vibrates much as a cantilevered beam about the constrained nodes. On the other hand, Figure 10.20b illustrates the mode shape for mode 2 oscillation. In mode 2, the structure exhibits an antisymmetric motion, in which the “halves” of the structure move in opposition to one another. Examination of the other modes reveals additional differences in the mode shapes.

Noting that Table 10.2 is, in fact, the modal matrix, it is a relatively simple matter to check the orthogonality conditions by forming the matrix triple products

$$[A]^T[M][A] = [I]$$
$$[A]^T[K][A] = \omega^2[I]$$

Within reasonable numerical accuracy, the relations are indeed true for this example. We leave the detailed check as an exercise.

10.12 PRACTICAL CONSIDERATIONS

The major problem inherent to dynamic structural analysis is the time-consuming and costly amount of computation required. In a finite difference technique, such as that represented by Equation 10.163, the system of equations must be solved at every time step over the time interval of interest. For convergence, the time step is generally quite small, so the amount of computation required is huge. In modal analysis, the burden is in computing natural frequencies and mode shapes. As practical finite element models can contain tens of thousands of degrees of freedom, the time and expense of computing all of the frequencies and mode shapes is prohibitive. Fortunately, to obtain reasonable approximations of dynamic response, it is seldom necessary to solve the full eigenvalue problem. Two practical arguments underlie the preceding statement. First, the lower-valued frequencies and corresponding mode shapes are more important in describing structural behavior. This is because the higher-valued frequencies most often represent vibration of individual elements and do not contribute significantly to overall structural response. Second, when structures are subjected to time-dependent forcing functions, the range of forcing frequencies to be experienced is reasonably predictable. Therefore, only system natural frequencies around that range are of concern in examining resonance possibilities.

Based on these arguments, many techniques have been developed that allow the computation (approximately) of a subset of natural frequencies and mode shapes of a structural system modeled by finite elements. While a complete discussion of the details is beyond the scope of this text, the following discussion explains the basic premises. (See Bathe [6] for a very good, rigorous description of the various techniques.) Using our notation, the eigenvalue problem that must be solved to obtain natural frequencies and mode shapes is written as

$$[K]\{A\} = \omega^2[M]\{A\} \tag{10.179}$$

The problem represented by Equation 10.179 is reduced in complexity by *static condensation* (or, more often, *Guyan reduction* [11]) using the assumption that all the structural mass can be lumped (concentrated) at some specific degrees of freedom without significantly affecting the frequencies and mode shapes of interest. Using the subscript a (active) to represent degrees of freedom of interest and subscript c (constrained) to denote all other degrees of freedom Equation 10.179 can be partitioned into

$$\begin{bmatrix} [K_{aa}] & [K_{ac}] \\ [K_{ca}] & [K_{cc}] \end{bmatrix} \begin{Bmatrix} \{A_a\} \\ \{A_c\} \end{Bmatrix} = \omega^2 \begin{bmatrix} [M_{aa}] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{A_a\} \\ \{A_c\} \end{Bmatrix} \tag{10.180}$$

In Equation 10.180, $[M_{aa}]$ is a diagonal matrix, so the mass has been lumped at the degrees of freedom of interest. The "constrained" degrees of freedom are constrained only in the sense that we assign zero mass to those degrees. The lower partition of Equation 10.180 is

$$[K_{ca}]\{A_a\} + [K_{cc}]\{A_c\} = \{0\} \tag{10.181}$$

and this equation can be solved as

$$\{A_c\} = -[K_{cc}]^{-1}[K_{ca}]\{A_a\} \tag{10.182}$$

to eliminate $\{A_c\}$. Substituting Equation 10.182 into the upper partition of Equation 10.180, we obtain

$$([K_{aa}] - [K_{ac}][K_{cc}]^{-1}[K_{ca}])\{A_a\} = \omega^2[M_{aa}]\{A_a\} \tag{10.183}$$

as the *reduced eigenvalue problem.* Note that all terms of the original stiffness matrix are retained but not those of the mass matrix. Another way of saying this is that the stiffness matrix is exact but the mass matrix is approximate.

The difficult part of this reduction procedure lies in selecting the degrees of freedom to be retained and associated with the lumped mass terms. Fortunately, finite element software systems have such selection built into the software. The user generally need specify only the number of degrees of freedom to be retained, and the software selects those degrees of freedom based on the smallest ratios of diagonal terms of the stiffness and mass matrices. Other algorithms are used if the user is interested in obtaining the dynamic modes within a specified frequency. In any case, the retained degrees of freedom are most often called *dynamic degrees of freedom* or *master degrees of freedom.*

This discussion is meant to be for general information and does not represent a hard and fast method for reducing and solving eigenvalue problems. Indeed, reference to Equation 10.182 shows that the procedure requires finding the inverse of a huge matrix to accomplish the reduction. Nevertheless, several powerful techniques have been developed around the general reduction idea. These include subspace iteration [12] and the Lanczos method [13]. The user of a particular finite element analysis software system must become familiar with the various options presented for dynamic analysis, as multiple computational schemes are available, depending on model size and user needs.

10.13 SUMMARY

The application of the finite element method to structural dynamics is introduced in the general context of linear systems. The basic ideas of natural frequency and mode shapes are introduced using both discrete spring-mass systems and general structural elements. Use of the natural modes of vibration to solve more-general problems of forced vibration is emphasized. In addition, the Newmark finite difference method for solving transient response to general forcing functions is developed. The chapter is intended only as a general introduction to structural dynamics. Indeed, many fine texts are devoted completely to the topic.

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PROBLEMS

10.1 Verify by direct substitution that Equation 10.5 is the general solution of Equation 10.4.

10.2 A simple harmonic oscillator has $m = 3$ kg, $k = 5$ N/mm. The mass receives an impact such that the initial velocity is 5 mm/sec and the initial displacement is zero. Calculate the ensuing free vibration.

10.3 The equilibrium deflection of a spring-mass system as in Figure 10.1 is measured to be 1.4 in. Calculate the natural circular frequency, the cyclic frequency, and period of free vibrations.

10.4 Show that the forced amplitude given by Equation 10.28 can be expressed as

$$U = \frac{X_0}{1 - r^2} \qquad r \neq 1$$

with $X_0 = F_0/k$ *equivalent static deflection* and $r = \omega_f/\omega \equiv$ *frequency ratio.*

10.5 Determine the solution to Equation 10.26 for the case $\omega_f = \omega$. Note that, for this condition, Equation 10.29 is *not* the correct solution.

10.6 Combine Equations 10.5 and 10.29 to obtain the complete response of a simple harmonic oscillator, including both free and forced vibration terms. Show that, for initial conditions given by $x(t = 0) = x_0$ and $\dot{x}(t = 0) = v_0$, the complete response becomes

$$x(t) = \frac{v_0}{\omega} \sin \omega t + x_0 \cos \omega t + \frac{X_0}{1 - r^2}(\sin \omega_f t - r \sin \omega t)$$

with X_0 and r as defined in Problem 10.4.

10.7 Use the result of Problem 10.6 with $x_0 = v_0 = 0$, $r = 0.95$, $X_0 = 2$, $\omega_f = 10$ rad/sec and plot the complete response $x(t)$ for several motion cycles.

10.8 For the problem in Example 10.2, what initial conditions would be required so that the system moved (a) in the fundamental mode only or (b) in the second mode only?

10.9 Using the data and solution of Example 10.2, normalize the modal matrix per the procedure of Section 10.7 and verify that the differential equations are uncoupled by the procedure.

10.10 Using the two-element solution given in Example 10.4, determine the modal amplitude vectors. Normalize the modal amplitude vectors and show that matrix product $[A]^T[M][A]$ is the identity matrix.

10.11 The 2 degrees-of-freedom system in Figure 10.4 is subjected to an external force $F_2 = 10 \sin 8t$ lb applied to node 2 and external force $F_3 = 6 \sin 4t$ lb applied to node 3. Use the normalized modal matrix to uncouple the differential equations and solve for the forced response of the nodal displacements. Use the numerical data of Example 10.2.

10.12 Solve the problem of Example 10.4 using two equal-length bar elements except that the mass matrices are *lumped;* that is, take the element mass matrices as

$$\left[m^{(1)}\right] = \left[m^{(2)}\right] = \frac{\rho A L}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

How do the computed natural frequencies compare with those obtained using consistent mass matrices?

10.13 Obtain a refined solution for Example 10.4 using three equal-length elements and lumped mass matrices. How do the frequencies compare to the two-element solution?

10.14 Considering the rotational degrees of freedom involved in a beam element, how would one define a lumped mass matrix for a beam element?

10.15 Verify the consistent mass matrix for the beam element given by Equation 10.78 by direct integration.

10.16 Verify the mass matrix result of Example 10.6 using Gaussian quadrature numerical integration.

10.17 Show that, within the accuracy of the calculations as given, the sum of all terms in the rectangular element mass matrix in Example 10.6 is twice the total mass of the element. Why?

10.18 What are the values of the terms of a lumped mass matrix for the element in Example 10.6?

10.19 Assume that the dynamic response equations for a finite element have been uncoupled and are given by Equation 10.120 but the external forces are not sinusoidal. How would you solve the differential equations for a general forcing function or functions?

10.20 Given the solution data of Example 10.7, assume that the system is changed to include damping such that the system damping matrix (after setting $u_1 = 0$) is given by

$$[C] = \begin{bmatrix} 2c & -c & 0 \\ -c & 2c & -c \\ 0 & -c & c \end{bmatrix}$$

Show that the matrix product $[A]^T[C][A]$ does not result in a diagonal matrix.

10.21 Perform the matrix multiplications indicated in Equation 10.177 to verify the result given in Equation 10.178.

10.22 For the truss in Example 10.10, reformulate the system mass matrix using lumped element mass matrices. Resolve for the frequencies and mode shapes using the finite element software available to you, if it has the lumped matrix available as an option (most finite element software includes this option).

10.23 If you formally apply a reduction procedure such as outlined in Section 10.12, which degrees of freedom would be important to retain if, say, we wish to compute only four of the eight frequencies?